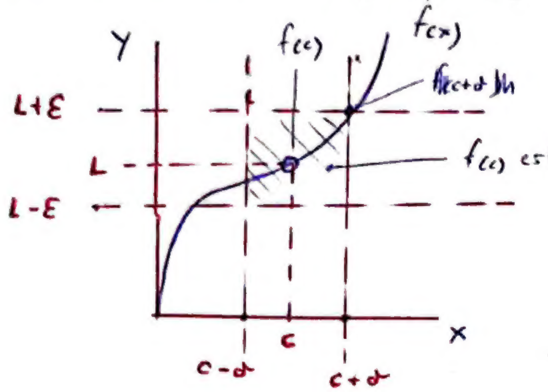
Definición formal de límite

- Supongamos que tenemos una función $f(x)$, y queremos calcular el límite de $f(x)$ cuando $x \to c$ ($c \in \mathbb{R}$). Decimos que el límite $\lim_{x \to c} f(x) = L$, $L \in \mathbb{R}$. Esto implica que $f(c) = L$ ($\lim_{x\to c} f(x) = L \Leftrightarrow f(c) = L$).
- ~~Para~~ Supongamos que L estuviera dentro de un rango de valores, por ejemplo, $L-\varepsilon < L < L+\varepsilon$, para algún $\varepsilon \in \mathbb{R}$. Entonces, existe un rango en x tal que $f(x) \in [L-\varepsilon, L+\varepsilon]$ y $c \in [L-\varepsilon, L+\varepsilon]$. Si al equivalente de ε en x lo llamamos δ, entonces $c-\delta < c < c+\delta$, y para todo x en éste rango hay un valor $y \in [L-\varepsilon, L+\varepsilon]$.



$f(c)$ está dentro de éste intervalo.

- $\varepsilon > 0$ y $\delta > 0$
- $f(c+\delta) \neq L+\varepsilon$ $(c+\delta, L+\varepsilon)$ es un punto y lo mismo para $(c-\delta, L-\varepsilon)$.
- Para cada $\varepsilon > 0$, existe un $\delta > 0$.

(Esto es lo que se pretende demostrar).

- Asumiendo que $f(c)$ esté definido y que $f(c) = L$, $f(c)$ es el centroide del área entre $L-\varepsilon$, $L+\varepsilon$, $c-\delta$ y $c+\delta$. Esto implica que si hacemos ε más pequeño, el punto $f(c)$ estará siempre dentro del área, sin importar lo arbitrariamente pequeño que sea ε, para cualquier valor $\delta > 0$. Los vértices del área ($f(c-\delta)$ y $f(c+\delta)$) no tienen que estar necesariamente dentro de la función. (Pero asumimos que lo estén en la teoría) Asumimos que $f(x)=L+\varepsilon$ está en la función
- Para encontrar el vértice $f(c+\delta)$ $L+\varepsilon$, buscamos un x tal que $f(x) = L+\varepsilon$. Sustrayendo L a ambos lados, obtenemos $\varepsilon = f(x) - L$. Y de aquí obtenemos que $x = c+\delta$, para $\delta > 0$.
- Buscamos los puntos x dentro del área $\boxed{-\varepsilon < f(x) - L < \varepsilon}$, ya que los bordes no nos interesan. O lo que es lo mismo, $|f(x) - L| < \varepsilon$. Si $x = c+\delta$, entonces $\delta = x - c$, para $\delta > 0$. O lo que es lo mismo; $0 < |x-c| < \delta$.

Esta es la clave Es decir, para todo ε en $|f(x) - L| < \varepsilon$, existe un δ tal que $0 < |x-c| < \delta$.

- Esto implica que al reducir el valor de ε, se reduce el valor de δ, y con ello se reduce el área centrada en $f(c)$. Si $f(c)$ está definido y $f(c) = L$, entonces $f(c)$ estará siempre dentro, no importa el valor de ε.
- Si $f(c) \neq L$, entonces $f(c)$ queda fuera del área en algún momento y la inigualdad no se cumple.
- Si $f(c)$ no está definido, entonces no hay un δ para cada ε, y la inigualdad no se cumple.

Las inigualdades deben cumplirse para todo $x \neq c$.

* $f(x) - L = 0$ si $\lim_{x\to c} f(x) = L$. Por tanto, para todo $\varepsilon > 0$, debe cumplirse que $-\varepsilon < f(x) - L < \varepsilon$. En tal caso, $(f(c), c)$ está dentro del área.

EJEMPLOS

$\boxed{\lim_{x\to 2} 2x = 4}$

Supongamos que ésto es cierto. Entonces $|2x-4| \le \varepsilon$, $\varepsilon > 0$. Resolviendo la inigualdad:

$$2|x-2|<\varepsilon \Rightarrow |x-2|<\frac{\varepsilon}{2} \text{ para todo } \delta>0 \text{ tal que } 0<|x-2|<\delta.$$

Sustituimos δ por $\frac{\varepsilon}{2}$:

$0<|x-2|<\frac{\varepsilon}{2}$ se cumple para todo $x \ne c$, o $x \ne 2$.

Multiplicando ambos lados por 2, obtenemos $0<2|x-2|<\varepsilon$. Es decir, es verdadero que para cada $\varepsilon>0$, existe un $\delta>0$, que es lo que se quería demostrar.

$\boxed{\lim_{x\to 2} 2x = 5}$

Supongamos que ésto es cierto. Entonces $|2x-5|<\varepsilon$, $\varepsilon>0$. Para todo $\varepsilon>0$, hay un $\delta>0$ tal que $0<|x-2|<\delta$. O lo que es lo mismo:

$$-\varepsilon<2x-5<\varepsilon \quad \text{o} \quad 5-\varepsilon<2x<\varepsilon+5 \quad \text{o} \quad \frac{5-\varepsilon}{2}<x<\frac{5+\varepsilon}{2}.$$

Sustituyendo δ por $\frac{5+\varepsilon}{2}$, $0<|x-2|<\frac{5+\varepsilon}{2}$ o $0<2|x-2|<5+\varepsilon$ o $-5<2|x-2|-5<\varepsilon$.

Pero $|2x-5| \ne 2|x-2|-5$ para todo $x\ne 2$. Por tanto, $f(2)=5$ queda fuera del área del límite para algún valor de $\varepsilon>0$. Por tanto, $\lim_{x\to 2} 2x \ne 5$.

$\boxed{\lim_{x\to 0} \frac{1}{x} = 0}$

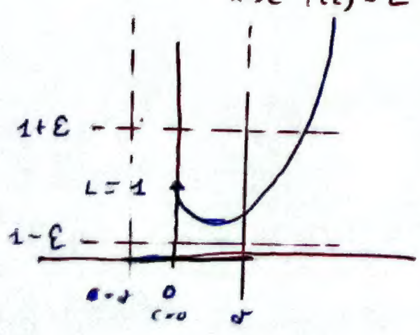
Supongamos que ésto es cierto. Entonces $\left|\frac{1}{x}\right|<\varepsilon$, $\varepsilon>0$, para algún valor $\delta>0$ tal que $0<|x|<\delta$. O lo que es lo mismo,

$-\varepsilon<\frac{1}{x}<\varepsilon$, o $-x\varepsilon<1<x\varepsilon$ si $0<|x|<\delta$. Sustituyendo δ por $x\varepsilon$, $0<|x|<x\varepsilon$. Ésta inigualdad no se cumple cuando $x<0$, $0<\varepsilon<1$ y $x\ge 1$, y por tanto no hay un $\delta>0$ para cada $\varepsilon>0$. Por tanto, el punto $f(c)$ no está definido, y el límite no existe.

$\boxed{\lim_{x\to 0} x^x = 1}$

Supongamos que ésto es cierto. Entonces $|x^x-1|<\varepsilon$, o $-\varepsilon<x^x-1<\varepsilon$, o $1-\varepsilon<x^x<1+\varepsilon$. Para cada $\varepsilon>0$, existe un $\delta>0$ tal que $0<|x|<\delta$. Sustituyendo δ por $1+\varepsilon$, $0<|x|<1+\varepsilon$. o $-1<|x|-1<\varepsilon$. Si $0<\varepsilon<1$, la inigualdad se cumple para todo x en $0<x\le 1$ $(x\ne 0)$, y para todo $x\ne 0$ si $\varepsilon>1$. Por tanto, el límite existe y $0^0=1$.

- La función $f(x)$ existe en un punto x, y está definida para ese punto, si el límite $\lim_{x\to c} f(c)=L$ existe.



Dado que $-1<|x|-1<\varepsilon$, también se cumple para todo x en $-1<x<0$ si $0<\varepsilon<1$, y para $x<-1$ si $\varepsilon>1$.

El límite, por tanto, está definido a ambos lados. Por tanto, el límite existe.

Ecuaciones vectoriales y paramétricas de una línea

Sean dos puntos P y Q en R^2 o R^3, la ecuación de la línea es

$\overrightarrow{PQ} = \langle Q_x - P_x, Q_y - P_y \rangle$ en R^2 y $\overrightarrow{PQ} = \langle Q_x - P_x, Q_y - P_y, Q_z - P_z \rangle$ en R^3

Si $Q = \langle x, y, z \rangle$ y $P = \langle x_0, y_0, z_0 \rangle$, la ecuación vectorial es

$$\overrightarrow{PQ} = (x - x_0)\hat{i} + (y - y_0)\hat{j} + (z - z_0)\hat{k}$$

La parametrización dependerá del punto que se tome como referencia. Sea t un escalar:

$r(t) = \langle x + (x - x_0)t, y + (y - y_0)t, z + (z - z_0)t \rangle$ tomando Q como referencia

$r(t) = \langle x_0 + (x - x_0)t, y_0 + (y - y_0)t, z_0 + (z - z_0)t \rangle$ tomando P como referencia

Ejemplo:

$P = (-3, 2, -3)$, $Q = (1, -1, 4)$

$\overrightarrow{PQ} = (1 - (-3))\hat{i} + (-1 - 2)\hat{j} + (4 - (-3))\hat{k} = 4\hat{i} - 3\hat{j} + 7\hat{k}$

$r(t) = \langle 1 + 4t, -1 - 3t, 4 + 7t \rangle$ Usando Q como referencia

$r(t) = \langle -3 + 4t, 2 - 3t, -3 + 7t \rangle$ Usando P como referencia.

Línea que pasa por un punto, paralela a un vector V.

Si P es un punto y V es un vector paralelo a la línea que pasa por P, entonces el vector de la línea tiene la misma dirección que V, y su longitud es un múltiplo de la longitud de V. La ecuación vectorial de la línea $r(t)$ es:

$r(t) = r_0 + tV$, $-\infty \leq t \leq \infty$ $\qquad r(t) = \langle x_0 + tV_1, y_0 + tV_2, z_0 + tV_3 \rangle$, $-\infty < t < \infty$

donde r_0 es el vector cuyo punto final es P.

Ejemplo:

$P = (-2, 0, 4)$ $\qquad V = 2i + 4j - 2k = \langle 2, 4, -2 \rangle$

$r(t) = \langle -2 + 2t, 4t, 4 - 2t \rangle$

Ecuación de un plano usando un punto P y un vector normal n en R^3

Si $P = (x, y, z)$ y n es normal al plano ($n = A\hat{i} + B\hat{j} + C\hat{k}$), entonces n es ortogonal a cualquiera de los vectores dentro del plano. Por tanto, si P_0 es un punto arbitrario que está dentro del plano, se debe cumplir que $n \cdot \overrightarrow{P_0P} = 0$. Sea $P_0 = (x_0, y_0, z_0)$,

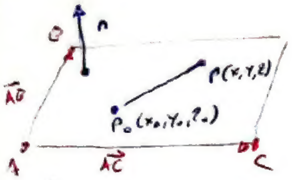
$n \cdot \overrightarrow{P_0P} = 0$ (1)

$\overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$

Sustituyendo en (1):

$$\boxed{A(x - x_0) + B(y - y_0) + C(z - z_0) = 0}$$

Ecuación de un plano



El plano que atraviesa $P_0(x_0,y_0,z_0)$ normal a $n = A\hat{i} + B\hat{j} + C\hat{k}$ tiene las propiedades:

Ecuación del vector: $n \cdot \vec{P_0P} = 0$

Ecuación de componentes: $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$

Ecuación simplificada: $Ax + By + Cz = D$ donde $D = Ax_0 + By_0 + Cz_0$

De modo que se puede obtener un plano con un punto y la normal n:

Encuentra la ecuación del plano que atraviesa $P_0(-3,0,7)$ perpendicular a $n = 5\hat{i} + 2\hat{j} - \hat{k}$

$$\underset{A}{5}(x - (\underset{x_0}{-3})) + \underset{B}{2}(y - \underset{y_0}{0}) + (\underset{C}{-1})(z - \underset{z_0}{7}) = 0$$

$$5x + 15 + 2y - z + 7 = 0, \quad \boxed{5x + 2y - z = -22}$$

O también con 3 puntos, computando la normal como el producto cruzado:

La ecuación del plano que atraviesa $A(0,0,1)$, $B(2,0,0)$, $C(0,3,0)$ es:

$\vec{AB} = 2\hat{i} + 0\hat{j} - \hat{k}$

$\vec{AC} = 0\hat{i} + 3\hat{j} - \hat{k}$

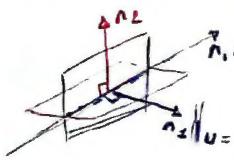
$$n = \underbrace{\det\left(\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & -1 \end{bmatrix}\right)}_{\vec{AB} \times \vec{AC}} = \underset{A}{3}\hat{i} + \underset{B}{2}\hat{j} + \underset{C}{6}\hat{k}$$

Usando A como P_0: $\boxed{3x + 2y + 6z = 6}$

Línea de intersección entre dos planos

Dos planos son paralelos si sus normales son paralelas, o si $n_1 = k n_2$ para ~~and~~ algún escalar k. Si no son paralelos, entonces intersectan entre sí.

La línea de intersección es paralela al producto cruzado de las normales $(n_1 \times n_2)$. Cualquier escalar múltiplo de $n_1 \times n_2$ servirá igual.



El vector $n_1 \times n_2$ puede parametrizarse en una línea, encontrando alguno de los puntos comunes en los planos.

Encuentra una ecuación paramétrica para la línea de intersección entre los planos $U = 3x - 6y - 2z$ y $V = 2x + y - 2z = 5$

$$n = U \times V = \det\left(\begin{bmatrix} 3 & -6 & -2 \\ 2 & 1 & -2 \end{bmatrix}\right) = \underset{n_x}{14}\hat{i} + \underset{n_y}{2}\hat{j} + \underset{n_z}{15}\hat{k}$$

Siendo $z = 0$, resolvemos para x e y:

$$3x - 6y = 15 \rightarrow \boxed{\begin{matrix} 3(5) - 6(0) = 15 \\ x=5 \quad y=0 \quad z=0 \end{matrix}} \rightarrow \boxed{\begin{matrix} 2x + y = 5 \\ 2(2) + (1) = 5 \\ x=2 \quad y=1 \quad z=0 \end{matrix}}$$

$$P_0 = (\underset{x_1 - x_0}{(5-2)}, \underset{y_1 - y_0}{(0-1)}, \underset{z_1 - z_0}{(0,0)}) = (\underset{x_0}{3}, \underset{y_0}{-1}, \underset{z_0}{0})$$

La ecuación paramétrica es:

$$\boxed{\begin{matrix} x = 3 + 14t & y = -1 + 2t & z = 15t \\ x_0 + n_x t & y_0 + n_y t & z_0 + n_z t \end{matrix}}$$

Distancia del punto S a un plano con normal n en el punto P

$$\boxed{d = \left|\vec{PS} \cdot \frac{n}{|n|}\right|}$$

Encuentra la distancia entre $S(1,1,3)$ y el plano $3x + 2y + 6z = 6$:

$n = 3\hat{i} + 2\hat{j} + 6\hat{k}$ (coeficientes del plano)

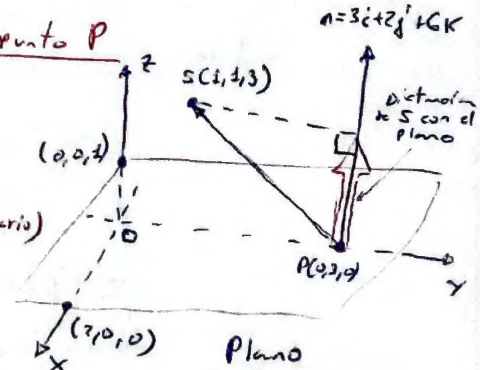
$P = (0,3,0)$ (Intersección Y con el plano) (Arbitrario)

$\vec{PS} = (1-0)\hat{i} + (1-3)\hat{j} + (3-0)\hat{k} = \hat{i} - 2\hat{j} + 3\hat{k}$

$|n| = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{49} = 7$

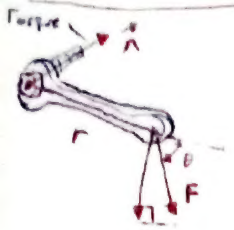
$$d = \left|\vec{PS} \cdot \frac{n}{|n|}\right| = \left|(\hat{i} - 2\hat{j} + 3\hat{k}) \cdot \left(\frac{3}{7}\hat{i} + \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k}\right)\right| =$$

$$= \left|\frac{3}{7} - \frac{4}{7} + \frac{18}{7}\right| = \boxed{\frac{17}{7}}$$



Ángulo entre dos planos

Producto punto de las normales del plano. Los coeficientes de los planos representan las componentes de las normales. $U \cdot V = |U||V|\cos\theta$, $\cos\theta = \frac{U \cdot V}{|U||V|}$ $\theta = \cos^{-1}\cos\theta$



$$\text{Vector torque: } r \times F = (|r||F| \sin\theta)\, n$$

$$\text{Magnitud del vector torque: } |r||F| \sin\theta$$

r = longitud del eje
F = Magnitud de la fuerza
θ = Ángulo de giro
n = Torque (Fuerza ejercida)

Triple producto escalar

Volúmen del paralelepípedo = Área de la base · Altura

$$\boxed{|u\times v|\,|w| \cos\theta} = \boxed{|(u\times v)\cdot w|} \quad \text{(Escalar)}$$

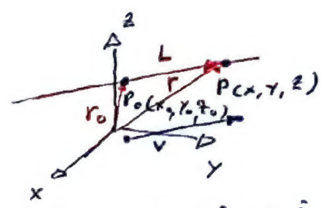
Triple producto escalar: $(u\times v)\cdot w = \det\left(\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\right) = u_1\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - v_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$ (Escalar)

Ecuación vectorial de una línea

- Línea L que pasa por $P_0(x_0, y_0, z_0)$ paralela a v:

$r(t) = r_0 + tv \quad -\infty < t < \infty$

r_0 y r son vectores de posición de P_0 y P respectivamente



Parametrización de la línea

- Parametrización de la línea L que pasa por $P_0(x_0, y_0, z_0)$ paralela a $v = v_1 i + v_2 j + v_3 k$

$x = x_0 + tv_1 \quad y = y_0 + tv_2, \quad z = z_0 + tv_3 \qquad -\infty < t < \infty$

t es el parámetro que define la longitud (en ambos casos)

- Se puede tratar la línea con: Punto P_0 y vector v paralelo, o con dos puntos P y Q que forman un vector $\vec{PQ} = Q_x - P_x i + Q_y - P_y j + Q_z - P_z k$, y cualquiera de los puntos P o Q como P_0.
- Se puede restringir t a un intervalo concreto para especificar un segmento (ej: $0 \le t \le 1$).

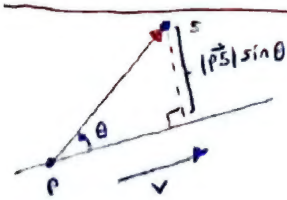
Movimiento

$$r(t) = \underset{\text{Posición inicial}}{r_0} + \underset{\text{Tiempo}}{t}\,\underset{\text{Velocidad}}{|v|}\,\underset{\text{Dirección}}{\frac{v}{|v|}} \quad \text{(Vector)}$$

$r_0 = \langle 0,0,0 \rangle$
$t = t$
Velocidad = 60 km/h
Dirección = $\langle 1,1,1 \rangle$

$r(t) = 0 + t(60)\left(\frac{1}{\sqrt{3}} i + \frac{1}{\sqrt{3}} j + \frac{1}{\sqrt{3}} k\right) =$
$= 20\sqrt{3}\, t\,(i + j + k)$
$r(10) = 200\sqrt{3}\,(i+j+k) = \langle 200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3} \rangle$

Distancia del punto S a la línea que atraviesa P paralela a v



$$|\vec{PS}| \sin\theta \rightarrow \frac{|\vec{PS}||v| \sin\theta}{|v|} = \frac{|\vec{PS} \times v|}{|v|} \quad \text{(Escalar)}$$

- Distancia entre el punto $S(1,1,5)$ y la línea L: $x = 1 + t$, $y = 3 - t$, $z = 2t$

$P_0 = (1, 3, 0)$
$v = i - j + 2k$

$\vec{PS} = (\overset{S_x\,x_0}{1-1})\,i + (\overset{S_y\,y_0}{1-3})\,j + (\overset{S_z\,z_0}{5-0})\,k = -2j + 5k$

$|\vec{PS} \times v| = \det\left(\begin{bmatrix} 0 & -2 & 5 \\ 1 & -1 & 2 \end{bmatrix}\right) = i + 5j + 2k$

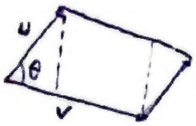
$$d = \frac{|\vec{PS} \times v|}{|v|} = \frac{\sqrt{1+25+4}}{\sqrt{1+1+4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}$$

Vectores: Producto cruzado

En $\mathbb{R}^2$, el producto cruzado es el área del paralelogramo.

$$u \times v = u_1 v_2 - v_1 u_2 = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = \det\left(\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right)$$ ~~(Escalar)~~ ~~(Vector)~~

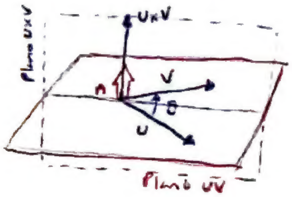
(Escalar cuando la determinante es cuadrada (2×2, 3×3...))
(Vector en cualquier otro caso (2×3, etc))



$$\boxed{A = |u \times v| = |u||v||\sin\theta||n| = |u||v|\sin\theta}$$ (Escalar)

En $\mathbb{R}^3$, el producto cruzado es un vector perpendicular al plano designado por los vectores u y v.

$$u \times v = \det\left(\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}\right) \text{ (vector)} \quad \triangle \quad \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

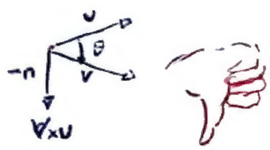


$$u \times v = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i + \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k \text{ (vector)}$$

$$u \times v = (u_2 v_3 - u_3 v_2) i + (u_1 v_3 - u_3 v_1) j + (u_1 v_2 - u_2 v_1) k \text{ (vector)}$$

$$u \times v = (|u||v|\sin\theta)\, n \text{ (vector)}$$

La dirección del ángulo θ determina la dirección del vector ortogonal



Propiedades

$\boxed{(ru) \times (sv) = (rs)(u \times v)}$

$\boxed{v \times u = -(u \times v)}$

$\boxed{0 \times u = 0}$

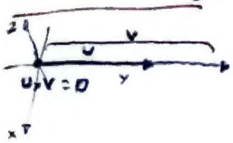
$\boxed{u \times (v + w) = u \times v + u \times w}$

$\boxed{(v + w) \times u = v \times u + w \times u}$

$\boxed{u \times (v \times w) = (u \cdot w)v - (u \cdot v)w}$

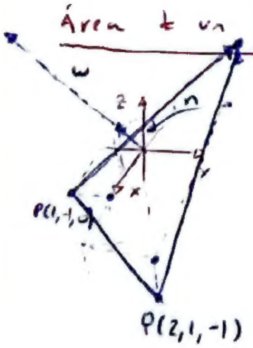
Propiedades geométricas

Cuando u y v son paralelos, es decir, su ángulo es $0, \pi, o\ n\pi\ (n \in \mathbb{Z})$, el producto cruzado es 0.



$$(|u||v|\sin\theta)\, n = (|u||v|\sin 0)\, n = (0)\, n = 0$$

Área de un triángulo y normal



$$\vec{PQ} = (2-1)i + (1+1)j + (-1-0)k = \boxed{i + 2j - k} \quad \boxed{|\vec{PQ}| = \sqrt{6}}$$

$$\vec{PR} = (-1-1)i + (1+1)j + (2-0)k = \boxed{-2i + 2j + 2k} \quad \boxed{|\vec{PR}| = \sqrt{12}}$$

$$w = \vec{PQ} \times \vec{PR} = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = (4+2)i + (2-2)j + 6k = \boxed{6i + 6k} \quad \boxed{|w| = \sqrt{36 \cdot 36} = \sqrt{2 \cdot 36} = 6\sqrt{2}}$$

$$\text{Área del triángulo} = \frac{1}{2}|w| = \frac{1}{2} 6\sqrt{2} = \boxed{3\sqrt{2}}$$

$$\text{Normal: } n = \frac{\vec{PQ} \times \vec{PR}}{|\vec{PQ} \times \vec{PR}|} = \frac{w}{|w|} = \frac{6}{6\sqrt{2}} i + \frac{6}{6\sqrt{2}} k = \boxed{\frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} k}$$

Centroide del triángulo

$$C = \begin{bmatrix} \dfrac{P_x + Q_x + R_x}{3} \\ \dfrac{P_y + Q_y + R_y}{3} \\ \dfrac{P_z + Q_z + R_z}{3} \end{bmatrix}$$

Vectores: Producto punto

Regla de los cosenos:

$c^2 = a^2 + b^2 - 2ab\cos\theta$

$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$ (Escalar)

$u \cdot v = |u||v| \cos\theta$ (Representación geométrica, escalar)

$|w|^2 = |u|^2 + |v|^2 - 2|u||v|\cos\theta$ (Aplicando regla de los cosenos, escalar)

Derivaciones:

$\theta = \cos^{-1}\left(\frac{u \cdot v}{|u||v|}\right)$ (Usando la repr. Geométrica, ángulo en radianes)

Propiedades

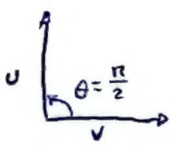
$\boxed{u \cdot v = v \cdot u}$ $\boxed{(cu)\cdot v = u \cdot (cv) = c(u \cdot v)}$

$\boxed{u \cdot (v + w) = u \cdot v + u \cdot w}$ $\boxed{u \cdot u = |u|^2}$

$\boxed{0 \cdot u = 0}$

Propiedades geométricas

Cuando el ángulo entre u y v es recto $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $\cos\theta = 0$. Por tanto, cuando el producto punto es 0, u y v son ortogonales.



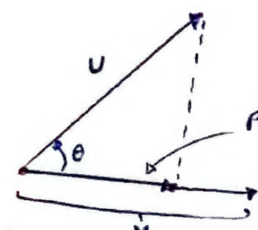
$$u \cdot v = |u||v| \cos\frac{\pi}{2} = |u||v|(0) = 0$$

Componente escalar de u en la dirección de v

$\boxed{|u|\cos\theta = \frac{u \cdot v}{|v|} = u \cdot \frac{v}{|v|}}$ (Escalar)

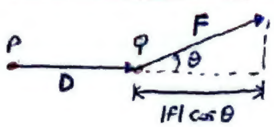
Proyección de u en v

$Proj_v u = \frac{u \cdot v}{|v|} \frac{v}{|v|} = \left(\frac{u \cdot v}{|v|^2}\right) v$ (vector)



Longitud de la proyección: $|u|\cos\theta$

Física: Trabajo



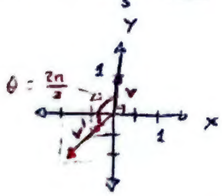
Siendo F una fuerza constante y D el desplazamiento de un objeto en $\overrightarrow{PQ}$

$W = (|F|\cos\theta)|D| = F \cdot D$ (Trabajo, Julios (J))

F en Newtons, θ en radianes o grados, D en metros.

Rotación de vector unitario en R^2

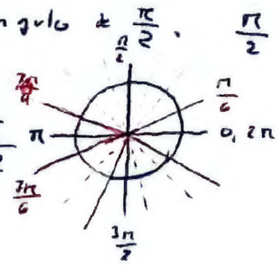
Rotar $\frac{2\pi}{3}$ rads el vector unitario $v = \langle 0, 1 \rangle$



v tiene un ángulo de $\frac{\pi}{2}$. $\frac{\pi}{2} + \frac{2\pi}{3} = \frac{3\pi + 4\pi}{6} = \frac{7\pi}{6}$

$\sin\frac{7\pi}{6} = -\frac{1}{2}$

$\cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$



$v' = \langle -\frac{\sqrt{3}}{2}, -\frac{1}{2} \rangle$

cos sin

Mean: $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \qquad \mu = \frac{1}{N} \sum_{i=1}^{N} x_i \qquad \Delta \bar{x} = \frac{1}{n} \Delta x_i$

Weighted mean $\bar{x}$: $\frac{(x_1 w_1) + (x_2 w_2) + \dots + (x_n w_n)}{w_1 + w_2 + \dots + w_n} = \left(\sum_{i=1}^{n} w_i \right)^{-1} \sum_{i=1}^{n} x_i w_i \qquad \Delta \bar{x} = \left(\sum w_i \right)^{-1} w_i \Delta w_i$

Median (middlemost value): $\begin{cases} \text{Odd} & \frac{x_{(n+1)}}{2} \\ \text{Even} & \frac{x_{(\frac{n}{2})} + x_{(\frac{n}{2}+1)}}{2} \end{cases}$ 50% Quantile

Variance (How the points spread from the mean)

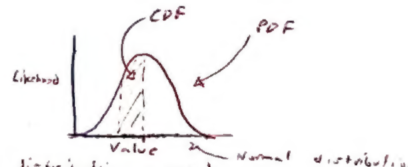
Population: $\sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2$ Sample: $s^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} (x_i - \bar{x})^2$

Increases variance

Standard Deviation STD: $\sqrt{\sigma^2} = \sigma$ or $\sqrt{s^2} = s$

$\Delta \sigma^2 = \frac{1}{N} 2(x_i - \mu)(1 - \frac{1}{N}) \Delta x_i \qquad \Delta s^2 = \frac{1}{n-1} 2(x_i - \bar{x})(1 - \frac{1}{n}) \Delta x_i$

$\Delta \sigma = \frac{1}{2(\sigma^2)^{\frac{1}{2}}} \Delta \sigma^2 \qquad \Delta s = \frac{1}{2(s^2)^{\frac{1}{2}}} \Delta s^2$



Normal distribution (where do the points gather)

Probability Density Function PDF (Creates a normal distribution curve)

$f_{(x)} = \frac{1}{\sigma} \sqrt{2\pi}\; e^{-\frac{1}{2}\left(\frac{x-\mu^2}{\sigma}\right)}$

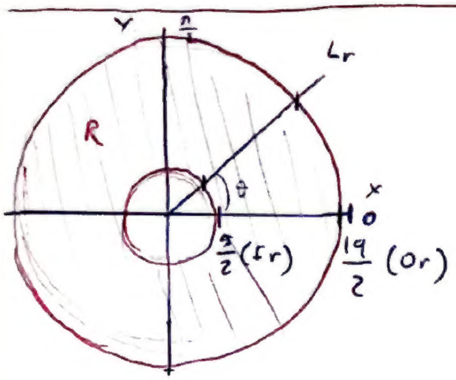
Probability between two values: Area under the normal distribution curve between values

CDF $= \int_I f_{(x)}\, dx$ I: Interval (values) (Cumulative distribution function)

$\boxed{0 \leq \int_I f_{(x)}\, dx \leq 1}$

Inverse CDF: Returns the value based on the probability

INTEGRAL TO FIND THE VOLUME OF A DISC



The region is a disc containing a small hole.

The equation for the disc (O_r) is $x^2+y^2-\frac{361}{4}=0$

The equation for the circle (I_r) is $x^2+y^2-\frac{25}{4}=0$

$O \Rightarrow x^2+y^2-\frac{361}{4}=0$

$I \Rightarrow x^2+y^2-\frac{25}{4}=0$

The radius of O (or) is $\frac{\frac{19}{2}}{2}=\frac{19}{4}$.

The radius of I (Ir) is $\frac{\frac{5}{2}}{2}=\frac{5}{4}$.

Geometrical Area

The area of O is: $A_O=\pi Or^2=\pi\left(\frac{19}{4}\right)^2=\pi\frac{361}{16}\approx 70'88218475 \text{ cm}^2$

The area of I is: $A_I=\pi Ir^2=\pi\left(\frac{25}{4}\right)^2=\pi\frac{25}{16}\approx 4'908738852$

The area of the surface of the disc is: $A_O-A_I=\frac{361\pi}{16}-\frac{25\pi}{16}=\frac{336\pi}{16}=\boxed{21\pi\approx 65'97344573}$

Integrating the region

The region is a disc, it's easier to use polar coordinates.

The region is the area bounded by the smaller circle and the larger circle ($I\leq r\leq O$). The radius r will cut I first then O. Doing the substitution

$x=r\cos\theta \quad y=r\sin\theta$

$\boxed{O=r^2\cos^2\theta+r^2\sin^2\theta-\frac{361}{4}=r^2-\frac{361}{4}}$, $\boxed{I=r^2-\frac{25}{4}}$

Both can also be expressed as the arcs:

$\boxed{O=r\cos\theta=\frac{19}{2}\cos\theta}$; $\boxed{I=r\cos\theta=\frac{5}{2}\cos\theta}$

In summary, region R is $\boxed{R:\ 0\leq\theta\leq\pi,\ \frac{5}{2}\cos\theta\leq r\leq\frac{19}{2}\cos\theta}$

The limit of the angle is from 0 to π since 0 and 2π are the same and would cancel each other.

The area of the disc is:

$$A=\iint_R r\,dA=\int_0^{\pi}\int_{\frac{5}{2}\cos\theta}^{\frac{19}{2}\cos\theta} r\,dr\,d\theta=\int_0^{\pi}\frac{r^2}{2}\Big]_{\frac{5}{2}\cos\theta}^{\frac{19}{2}\cos\theta}d\theta=\int_0^{\pi}\frac{\left(\frac{19}{2}\cos\theta\right)^2-\left(\frac{5}{2}\cos\theta\right)^2}{2}\,d\theta=$$

$$=\int_0^{\pi}\frac{361-25}{8}\cos^2\theta\,d\theta=\int_0^{\pi}42\cos^2\theta\,d\theta=42\left[\frac{\theta}{2}+\frac{\sin\theta\cos\theta}{2}\right]_0^{\pi}=42\frac{\pi}{2}=\boxed{21\pi\approx 65'97344573}$$

$A=21\pi$

SUBSTITUTION

$$\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\, dx\, dy \qquad \text{Let } f(x,y) = \frac{2x-y}{2}$$

~~- Find x, y~~

$$\cancel{\frac{2x-y}{2} = \frac{2x}{x} - \frac{y}{2} = x - \frac{y}{2} \rightarrow \boxed{x = \frac{y}{2}} \rightarrow \boxed{y = 2x}}$$

- Let $u = \frac{2x-y}{2}$, $v = \frac{y}{2}$ $\left(\frac{2x-y}{2} = 0 \text{ when } x = \frac{y}{2} \right)$, Find x, y in terms of u, v

$v = \frac{y}{2}$, $\boxed{y = 2v}$

Substituting $y = 2v$: $u = \frac{2x-2v}{2} \rightarrow 2u = 2x - 2v \rightarrow u = x - v \rightarrow \boxed{x = u+v}$

- Find the limits for G

$0 \le y \le 4$ in R. When $y=0$, $v = \frac{0}{2} = 0$ in G. when $y = 4$, $v = \frac{4}{2} = 2$ in G.

$\frac{y}{2} \le x \le \frac{y}{2}+1$ in R. When $x = \frac{y}{2}$, $u = \frac{y+2-y}{2} = 1$. When $x = \frac{y}{2}+1$, $u = \frac{y-y}{2} = 0$.

$$\boxed{G:\ 0 \le v \le 2,\ 0 \le u \le 1}$$

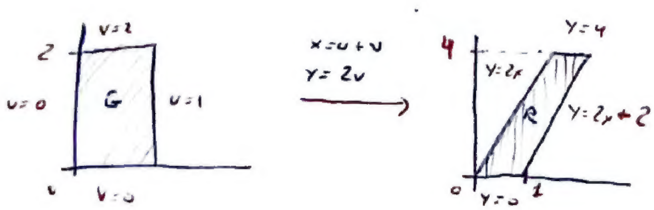
- Find the Jacobian transformation $J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$

Being $x = u+v$, $y = 2v$

$$J(u,v) = \begin{vmatrix} \frac{\partial}{\partial u}(u+v) & \frac{\partial}{\partial v}(u+v) \\ \frac{\partial}{\partial u}(2v) & \frac{\partial}{\partial v}(2v) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$

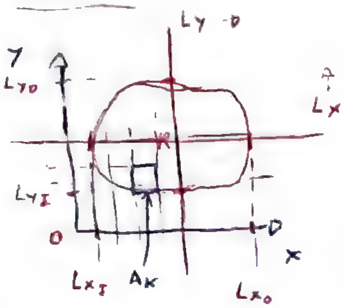
- Apply $\iint_R f(x,y)\, dx\, dy = \iint_G f(g(u,v), h(u,v))\, |J(u,v)|\, du\, dv$

$$\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\, dx\, dy = \int_0^2 \int_0^1 u\, |J(u,v)|\, du\, dv = \int_0^2 \int_0^1 2u\, du\, dv = \int_0^2 dv = 2$$



INTEGRATION

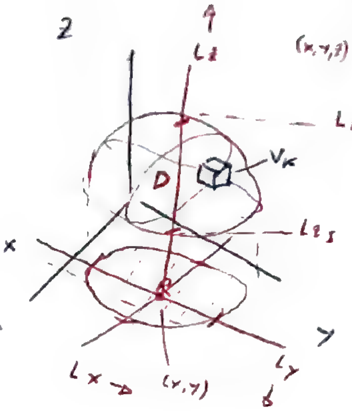
CARTESIAN



$\boxed{\Delta A_K = \Delta x_K \Delta y_K}$ $\Delta y_K \Delta x_K$

$$\iint_R f_{(x,y)}\, dA \qquad dA = dx\,dy \text{ or } dy\,dx$$

$L_{X_I} \leq x \leq L_{X_0}$ $L_{Y_I} \leq y \leq L_{Y_0}$



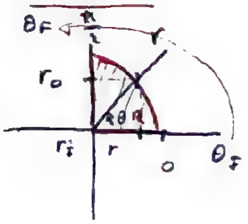
$\boxed{V_K = \Delta z_K \Delta y_K \Delta x_K}$

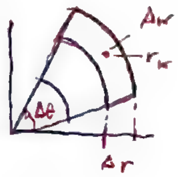
$$\iiint_D dV \qquad \iiint_D f_{(x,y,z)}\, dV$$

$dV = dz\,dy\,dx$ or any combination

$L_{Z_I} \leq z \leq L_{Z_0}$

POLAR



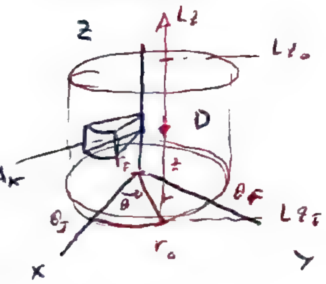


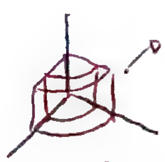
$\Delta A_K = r_K \Delta r \Delta\theta$ $r_I \leq r \leq r_0$

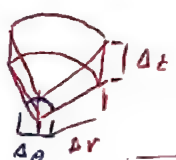
$dA = r\,dr\,d\theta$ $\theta_I \leq \theta \leq \theta_F$

$$\boxed{\iint_R f(r,\theta)\, r\,dr\,d\theta}$$

CYLINDRICAL







$\Delta A_K = \Delta z\, r \Delta r \Delta\theta$

$dV = dz\, r\, dr\, d\theta$

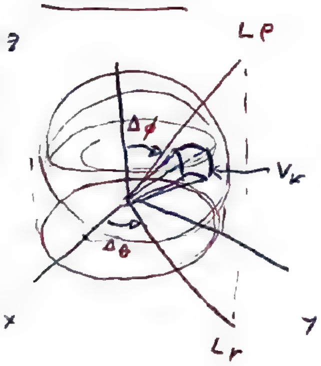
$\theta_I \leq \theta \leq \theta_F$

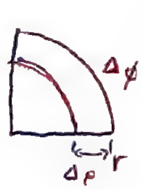
$r_I \leq r \leq r_0$

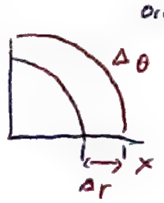
$z_I \leq z \leq z_0$

$$\boxed{\iiint_D f(r,\theta,z)\, dz\, r\, dr\, d\theta}$$

SPHERICAL







$\cancel{V_K} = r\Delta\theta\, \rho\Delta\phi\, \Delta r\Delta\theta$; $r = \rho\sin\phi$

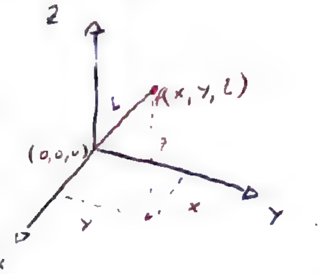
$\cancel{V_K} = \rho\Delta\phi\, \rho\sin\phi\, \Delta r = \rho^2 \sin\phi\, \Delta\phi\, \Delta r\, \Delta\theta$

$$\boxed{dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}$$

$$\boxed{\iiint_D f(\rho,\phi,\theta)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}$$

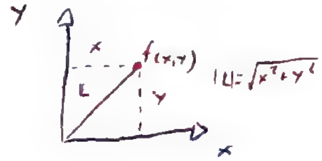
Coordinate systems

CARTESIAN

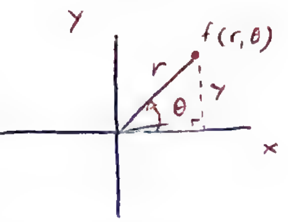


$$L = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$|L| = \sqrt{x^2 + y^2 + z^2}$$



POLAR COORDINATES

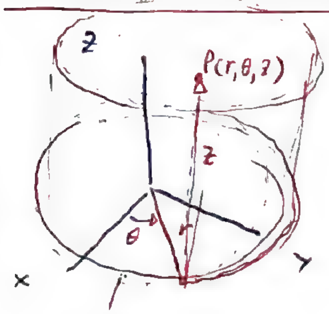


$\sin\theta = \dfrac{y}{r}$ $\quad x = r\cos\theta$ $\quad r = \dfrac{x}{\cos\theta} = \dfrac{y}{\sin\theta}$

$\cos\theta = \dfrac{x}{r}$ $\quad y = r\sin\theta$ $\quad \theta = \arcsin\left(\dfrac{y}{r}\right) = \arccos\left(\dfrac{x}{r}\right)$

$r = f(r,\theta) = \sqrt{x^2 + y^2}$

CILINDRICAL COORDINATES



$r^2 = x^2 + y^2$

$\boxed{x = r\cos\theta}$

$\boxed{y = r\sin\theta}$

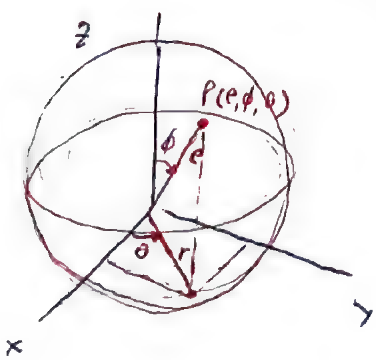
$\boxed{z = z}$

$\boxed{r^2 = x^2 + y^2} = r^2$

$$\boxed{\frac{y}{x} = \frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta} = \tan\theta}$$

$$r^2 = r^2\sin^2\theta + r^2\cos^2\theta = r^2(\sin^2\theta + \cos^2\theta) = r^2$$

SPHERICAL COORDINATES



$0 \le \phi \le \pi$

$0 \le \rho \le \infty$

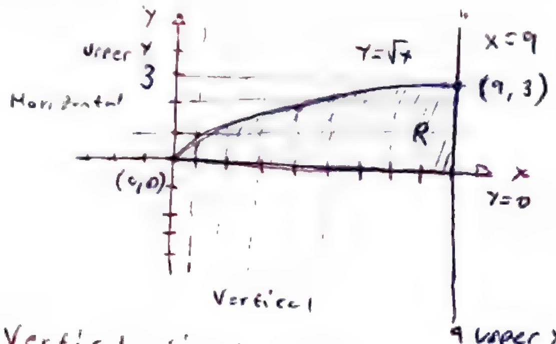
$\sin\phi = \dfrac{r}{\rho}$ $\quad \boxed{r = \rho\sin\phi}$

$\cos\phi = \dfrac{z}{\rho}$ $\quad \boxed{z = \rho\cos\phi}$

$$\boxed{\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}}$$

$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$

Area of integration (R) $\iint_R f(x,y)\,dA$



- R is bounded by $y=\sqrt{x}$, $y=0$ and $x=9$
- R is the area between those 3 curves.
- The crossing points are:

$\sqrt{x}=0$; $x=0$ $y=0$ $(0,0)$

$y=\sqrt{x}$; $x=9$; $y=\sqrt{9}=3$ $(9,3)$

Vertical slices! Those slices are perpendicular to the x-axis. For each x, there is a slice crossing each curve. The first curve is $y=0$, and the last one is $y=\sqrt{x}$. The partition is between: $\boxed{0\le x\le 9}$ and $\boxed{0\le y\le\sqrt{x}}$.

Horizontal slices: Those slices are parallel to the x-axis. For each y, there is a slice crossing each curve. The first curve sliced is $y=\sqrt{x}$. The last is $x=9$. We solve $y=\sqrt{x}$ for x! $\boxed{x=y^2}$. Since this is the first curve sliced, this is the lower bound for x.

The partition is between: $\boxed{0\le y\le 3}$ and $\boxed{y^2\le x\le 9}$.

Integrals: $\boxed{\int_0^9\int_0^{\sqrt{x}} f(x,y)\,dy\,dx}$ (Vertical) or $\boxed{\int_0^3\int_{y^2}^9 f(x,y)\,dx\,dy}$ (Horizontal)

R bounded by $y=e^{-x}$, $y=1$ and $x=\ln 3$

1. Calculate the crossing points

~~Lower~~ Bounds for x: 0, $\ln 3$

Bounds for y: $\frac{1}{3}$, 1

- $e^{-x}=1$; $x=0$ $\boxed{(0,1).}$
- Using $x=\ln 3$, $y=e^{-\ln 3}=\frac{1}{e^{\ln 3}}=\frac{1}{3}$ $\boxed{(\ln 3,\frac{1}{3})}$

Vertical slices; x is bounded between $x=0$ and $x=\ln 3$. The upper bound for 'y' is 1, and the lower bound is e^{-x} because $e^{-x}\le 1$ for all x.

$\boxed{R:\ 0\le x\le\ln 3,\ e^{-x}\le y\le 1}$

Horizontal slices! The upper bound for y is 1 and the lower bound for y is $\frac{1}{3}$. The upper bound for x is $\ln 3$. We need to calculate the lower bound for x;

Using $y=e^{-x}$; $\boxed{x=-\ln y}$

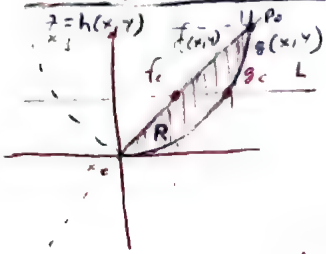
R: $\boxed{\frac{1}{3}\le y\le 1}$, $\boxed{-\ln y\le x\le\ln 3}$

Integrals: $\boxed{\int_0^{\ln 3}\int_{e^{-x}}^{1} f(x,y)\,dy\,dx}$ and $\boxed{\int_{\frac{1}{3}}^{1}\int_{-\ln y}^{\ln 3} f(x,y)\,dx\,dy}$

For vertical slices (dy,dx): Find the y limits first, then the x limits

For horizontal slices (dx,dy): Find the x limits first, then the y limits

Finding limits of double integrations



For a function $z = h(x,y)$, the area of the region R

$$\iint_R h(x,y)\, dA$$

where R: $\displaystyle\int_0^{x} \int_{g(x,y)}^{f(x,y)} h(x,y)\, dA$

Is cut through a line L: $\dfrac{g c_y - f c_y}{g c_x - f c_x}$. L can be horizontal or vertical depending on wether ~~we~~ we are integrating with respect to x or y first.

f_c is obtained solving ~~for~~ $f(x,y)$ for y first, and so is g_c for $g(x,y)$.

The new integral will be $\displaystyle\int_{x_0}^{x_1} \int_{g_c}^{f_c} h(x,y)\, dA$.

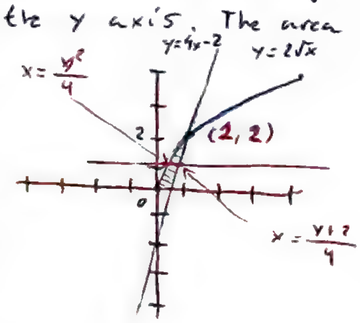
$z = 16 - x^2 - y^2$ bounded by $y = 2\sqrt{x}$, $y = 4x - 2$, and the x-axis

$$\iint_R (16 - x^2 - y^2)\, dA$$

① Determine the order of integration (x or y first). Since the bounds are in terms of y, y varies from $y=0$ to $y = 2\sqrt{x}$ for $0 \le x \le \frac{1}{2}$ and from $y = 2\sqrt{x}$ to $y = 4x-2$ for $\frac{1}{2} \le x \le 1$. Two double integrals would be required. Thus, we integrate with respect to x first.

$$\iint_R (16 - x^2 - y^2)\, dx\, dy$$

② Determine the region R. We are ~~derivatin~~ Integrating with respect to x, L will cut through the y axis. The area R is bounded by the functions $y = 2\sqrt{x}$ and $y = 4x - 2$.



For $y = 2\sqrt{x}$, $x = \frac{y^2}{4}$. For $y = 4x - 2$, $x = \frac{y+2}{4}$. L will cross these functions at these x-points. The new integral will be:

$$\boxed{\int_0^2 \int_{\frac{y^2}{4}}^{\frac{y+2}{4}} (16 - x^2 - y^2)\, dx\, dy}$$

③ Integrate.

$$\int_0^2 \left[16x - \frac{x^3}{3} - xy^2\right]_{\frac{y^2}{4}}^{\frac{y+2}{4}} dy = \int_0^2 \left[4(y+2) - \frac{(y+2)^3}{3(64)} - \frac{(y+2)y^2}{4} - 4y^2 + \frac{y^6}{3(64)} + \frac{y^4}{4}\right] dy =$$

$$= \frac{191y}{24} + \frac{63y^2}{32} - \frac{145y^3}{96} - \frac{49y^4}{768} + \frac{y^5}{20} + \frac{y^7}{1344}\Big|_0^2 = \frac{20803}{1680} \approx \boxed{12'4}$$

Note: To find the crossing points between the bounds $f(x,y)$ and $g(x,y)$, say that $f(x,y) = g(x,y)$ and solve for x.

$2\sqrt{x} = 4x - 2$

$2\sqrt{x} - 4x = -2$

$x = 1$ (by inspection)

Using $y = 2\sqrt{x}$: $2\sqrt{1} = 2(1) = 2$,

The crossing point is $(1, 2)$, so y ranges from $y=0$ (origin, x-intercept of the x-axis), to $y = 2$.

LOCAL MAXIMA AND MINIMA EXAMPLE

- Find the local extreme values of the function

$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

Calculate f_x and f_y

$\boxed{f_x = y - 2x - 2}$ $\boxed{f_y = x - 2y - 2}$

The extreme values happen when $f_x = 0$ and $f_y = 0$, so:

$$-2x + y = 2$$
$$x - 2y = 2$$

The solution of this system is $x = -2$, $y = -2$, since $-2x + y = x - 2y$.
Therefore a potential critical point is $\boxed{(-2,-2)}$.

- To check if this is a critical point or a saddle point, we'll perform a second derivative test. We need f_{xx}, f_{yy} and f_{xy}:

$\boxed{f_{xx} = -2}$ $\boxed{f_{yy} = -2}$ $\boxed{f_{xy} = 1}$

The discriminant (Hessian) is:

$$f_{xx} f_{yy} - f_{xy}^2 = -2(-2) - (1)^2 = 4 - 1 = 3$$

Because $f_{xx} f_{yy} - f_{xy}^2 > 0$, this is a critical point. And because $f_{xx} < 0$, this critical point is a local maximum. The value of f at this point is:

$$\boxed{f_{(-2,-2)} = -2(-2) - (-2)^2 - (-2)^2 - 2(-2) - 2(-2) + 4 = 8}$$

- Find the absolute maximum and minimum values of $f(x,y) = 2 + 2x + 4y - x^2 - y^2$ on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$, $y = 9 - x$.

a) Interior Points (Function)

$f_x = 0 \Leftrightarrow x = 1$ $\quad f_y = 0 \Leftrightarrow y = 2$

$f_x = 2 - 2x = 0$ $\quad f_y = 4 - 2y = 0$. Critical points: $\{(1,2)\}$. $\boxed{f_{(1,2)} = 7}$

b) Boundary points (Boundary triangle) $\quad 2 + 2x + 4(0) - x^2 - (0)^2$

- Segment OA ($y = 0$): $f(x,y) = f(x,0) = 2 + 2x - x^2$, $\overset{a}{0} \le x \le \overset{b}{9}$. Extremes: $\begin{cases} f(\overset{a}{0},0) = 2 \\ f(\underset{b}{9},0) = -61 \end{cases}$

Critical points: $f'(x,0) = 2 - 2x = 0 \Leftrightarrow x = 1$. $\overset{crit}{f}(1,0) = 3$.

- Segment OB, $x = 0$, $f(x,y) = f(0,y) = 2 + 4y - y^2$, $0 \le x \le 9$. Extreme values occur at $f'(0,y) = 0$. $f'(0,y) = 4 - 2y = 0 \Leftrightarrow y = 2$. $\overset{crit}{f}(0,2) = 6$ $\quad f(0,\overset{a}{0}) = 2$, $f(0,\underset{b}{9}) = -43$

- Segment AB. $y = 9 - x$:

$f(x,y) = 2 + 2x + 4(9 - x) - x^2 - (9 - x)^2 = -43 + 16x - 2x^2$

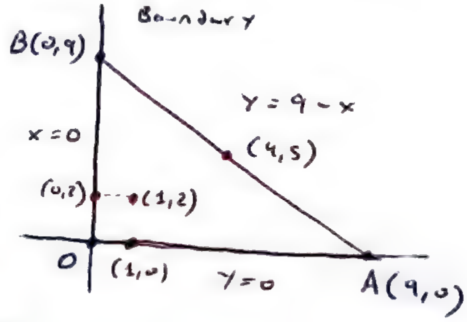
$f'(x, 9 - x) = 16 - 4x = 0$; $x = 4$

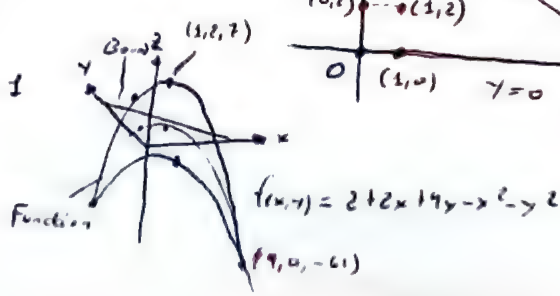
$y = 9 - 4 = 5$ $\quad f(x,y) = f(4,5) = -11$

- Summary: Candidates: $\{-61, -43, -11, 2, 3, 6, 7\}$.

Maximum value: 7 $\quad f(1,2) = 7$.

Minimum value: -61 $\quad f(9,0) = -61$





- $f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$

• Interior Points

$\boxed{f_x = 2x + y + 3}$ $\boxed{f_y = x + 2y - 3}$

[illegible]

Critical points occur when $f_x = f_y = 0$

System of eq

$2x + y + 3 = 0$

$x + 2y - 3 = 0$ $\quad R_1 + (-2)R_2 = -3y + 9 = 0$. $\boxed{y = \frac{9}{3} = 3}$

Substituting y in R_1: $2x + 3 + 3 = 0$; $2x = -6$; $\boxed{x = -\frac{6}{2} = -3}$

$\boxed{\text{Critical points: } \{(-3, 3)\}}$

• Boundary Points

$f(x,0) = x^2 + 3x + 4$, $f'(x,0) = 2x + 3$, $f'(x,0) = 0 \iff 2x + 3 = 0$; $x = -\frac{3}{2}$

$f(0,y) = y^2 - 3y + 4$; $f'(0,y) = 2y - 3$; $f'(0,y) = 0 \iff 2y = 3$; $y = \frac{3}{2}$

$f(-\frac{3}{2}, \frac{3}{2}) = (-\frac{3}{2})^2 + (-\frac{3}{2})\frac{3}{2} + (\frac{3}{2})^2 + 3(-\frac{3}{2}) - 3\frac{3}{2} + 4 = \cancel{\frac{9}{4}} - \cancel{\frac{9}{4}} + \frac{9}{4} + \cancel{\frac{9}{2}} - \cancel{\frac{9}{2}} + 4 = \frac{25}{4}$

However, $f(x,0) \to \infty$ as $x \to \infty$ and $f(0,y) \to \infty$ as $y \to \infty$.

Point $(-3, 3)$ is a local minimum. $f(-3,3) = -5 < \frac{25}{4} < \infty$

• Using the second derivative test

$\boxed{f_{xx} = 2}$, $\boxed{f_{yy} = 2}$, $\boxed{f_{xy} = 1}$

$\boxed{f_{xx} f_{yy} - (f_{xy})^2 = 2(2) - 1^2 = 3}$

The discriminant is > 0, so $(-3, 3)$ is a critical point.

$f_{xx} = 2 > 0$, $(-3, 3)$ is a local minimum. $f(-3,3) = -5$.

LOCAL MAXIMA AND MINIMA

- $f(a,b)$ is a local maximum value of f if $f(a,b) \geq f(x,y)$ for all domain points (x,y) in an open disc centered at (x,y)
- $f(a,b)$ is a local minimum value of f if $f(a,b) \leq f(x,y)$ for all domain points (x,y) in an open disc centered at (x,y).

FIRST DERIVATIVE TEST

- If $f(x,y)$ has a local maximum or minimum value at an interior point (a,b) of it's domain and if the first partial derivatives exists there, then

$$f_x(a,b) = 0 \quad \text{and} \quad f_y(a,b) = 0$$

- An interior point of the domain of a function $f(x,y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a critical point of f.

SADDLE POINT (PUNTO DE SILLA)

- A differenciable function $f(x,y)$ has a saddle point at a critical point (a,b) if in every open disk centered at (a,b) there are domain points (x,y) where $f(x,y) > f(a,b)$ and domain points (x,y) where $f(x,y) < f(a,b)$.
  The corresponding point $(a,b,f(a,b))$ on the surface $z = f(x,y)$ is called a saddle point of the surface.

SECON DERIVATIVE TESTS

Local maximum if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b)

Local minimum if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b)

Saddle point if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a,b)

Test inconclusive if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a,b)

Discriminant (Hessian): $f_{xx}f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$

ABSOLUTE MAXIMA AND MINIMA ON CLOSED BOUNDED REGIONS

1. List the interior points of R where f may have local maxima and minima and evaluate f at these points.
2. List the boundary points of R where f has a local maxima and minima and evaluate f at these points.
3. Look through the lists for the maximum and minimum values of f. These will be the absolute maximum and minimum values of f on R.

52. VARIATION IN ELECTRICAL RESISTANCE

The resistance R produced by wiring resistors of R_1 and R_2 ohms in parallel can be calculated from the formula $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.

a) Show that $dR = \left(\frac{R}{R_1}\right)^2 dR_1 + \left(\frac{R}{R_2}\right)^2 dR_2$

b) You have designed a two-resistor circuit

$R_1 = 100\,\Omega$

$R_2 = 400\,\Omega$

but there is always some variation in manufacturing and the resistors received by your firm will probably not have these exact values.
Will the value of R be more sensitive to variations in R_1 or R_2? Give reasons for your answer.

---

a) The formula for the resistance R is

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \boxed{\frac{R_1 R_2}{R_1 + R_2} = R}$$

The variation in R is:

$$dR = \frac{\partial R}{\partial R_1} dR_1 + \frac{\partial R}{\partial R_2} dR_2 = \frac{R_2^2}{(R_1+R_2)^2} dR_1 + \frac{R_1^2}{(R_1+R_2)^2} dR_2 = \left(\frac{R_2}{R_1+R_2}\right)^2 dR_1 + \left(\frac{R_1}{R_1+R_2}\right)^2 dR_2 =$$

$$= \left(\frac{R_2}{R_1+R_2}\,\frac{R_1}{R_1}\right)^2 dR_1 + \left(\frac{R_1}{R_1+R_2}\,\frac{R_2}{R_2}\right)^2 dR_2 = \left(\frac{R_1 R_2}{R_1(R_1+R_2)}\right)^2 dR_1 + \left(\frac{R_1 R_2}{R_2(R_1+R_2)}\right)^2 dR_2 =$$

$$= \left(\frac{\frac{R_1 R_2}{R_1+R_2}}{R_1}\right)^2 dR_1 + \left(\frac{\frac{R_1 R_2}{R_1+R_2}}{R_2}\right)^2 dR_2 = \boxed{\left(\frac{R}{R_1}\right)^2 dR_1 + \left(\frac{R}{R_2}\right)^2 dR_2}$$

b) Plugging the values of R_1 and R_2 into this formula, we get:

$$R = \frac{100 \cdot 400}{100+400} = \frac{40000}{500} = \frac{400}{5} = 80\,\Omega \text{ (total resistance } R\text{)}$$

$$dR = \left(\frac{80}{100}\right)^2 dR_1 + \left(\frac{80}{400}\right)^2 dR_2 = \left(\frac{4}{5}\right)^2 dR_1 + \left(\frac{1}{5}\right)^2 dR_2 = \boxed{\frac{16}{25} dR_1 + \frac{1}{25} dR_2}$$

Because $\frac{\partial R}{\partial R_2} < \frac{\partial R}{\partial R_1}$, the value of R will be more sensitive to changes in the value of R_1 than it will be to changes in the value of R_2.
However, this is only true when the difference between R_1 and R_2 is less than 16. That is, $R_2 < 16 R_1$.
When $R_2 > 16 R_1$, then R will be more sensitive to changes in R_2.
For this specific case, $400 < 1600$, which means the former statement applies.

EQUATION PLANE TANGENT TO NORMAL

$P_0(x_0, y_0, z_0)$: Point

$P(x,y,z)$: Point in the plane (arbitrary)

$n(ai + bj + ck)$: Normal of the plane (orthogonal to $\vec{P_0P}$)

$\vec{P_0P} = (x - x_0)i + (y - y_0)j + (z - z_0)k$

$\boxed{n \cdot \vec{P_0P} = 0}$

$$(ai + bj + ck) \cdot [(x - x_0)i + (y - y_0)j + (z - z_0)k] = 0$$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

EQUATION GRADIENT OF FUNCTION (2+ VARS)

$$f = f(x, y, \dots, n) \qquad \boxed{\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \dots + \frac{\partial f}{\partial n} v}$$

∇ ∇f grad

del LaTeX: \nabla

∇f is the direction of maximum rate of change.

DIRECTIONAL DERIVATIVES (Applies to 2+ vars (Append vars))

$$\left(\frac{df}{ds}\right)_{u, P_0} = \left.\frac{\partial f}{\partial x}\right|_{P_0} \frac{dx}{ds} + \left.\frac{\partial f}{\partial y}\right|_{P_0} \frac{dy}{ds} =$$

$f = f(x,y)$

$u = u_1 i + u_2 j$

$P_0 = (x_0, y_0)$

$x = x_0 + s u_1 \quad y = y_0 + s u_2$

$$= \left.\frac{\partial f}{\partial x}\right|_{P_0} u_1 + \left.\frac{\partial f}{\partial y}\right|_{P_0} u_2 =$$

$$= \left[\left.\frac{\partial f}{\partial x}\right|_{P_0} i + \frac{\partial f}{\partial y} j\right] \cdot [u_1 i + u_2 j] = \boxed{\nabla f|_{P_0} \cdot u}$$

$$\boxed{D_u f|_{P_0} = \nabla f|_{P_0} \cdot u}$$

Since $|\nabla f||u| \cos\theta = \nabla f \cdot u$,

- f increases more rapidly when $\cos\theta = 1$ $(\theta = 0)$. This is, u is in the direction of ∇f
- f decreases more rapidly when $\cos\theta = -1$ $(\theta = \pi)$. This is the direction opposite of ∇f $(-\nabla f)$.
- f has a zero change when $\cos\theta = 0$ $(\frac{\pi}{2}, \frac{3\pi}{2} \dots)$. This is any orthogonal direction to ∇f.

GRADIENT ALGEBRA

$\nabla(f + g) = \nabla f + \nabla g$

$\nabla(f - g) = \nabla f - \nabla g$

$\nabla(kf) = k \nabla f \quad k \in \mathbb{R}$

$\nabla(fg) = f \nabla g + g \nabla f$

$$\nabla\left(\frac{f}{g}\right) = \frac{g \nabla f - f \nabla g}{g^2}$$

CHAIN RULE FOR PATHS

$$\boxed{\frac{d}{dt} f(r(t)) = \nabla f(r(t)) \cdot r'(t)}$$

Let $r(t) = x(t) i + y(t) j + z(t) k$, $w = f(r(t))$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\frac{dw}{dt} = \underbrace{\left(\frac{\partial w}{\partial x} i + \frac{\partial w}{\partial y} j + \frac{\partial w}{\partial z} k\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k\right)}_{r'(t)}$$

$$\frac{dw}{dt} = \nabla f(r(t)) \cdot r'(t)$$

TANGENT PLANE TO A LEVEL SURFACE

P has a normal equal to ∇f

$P_0(x_0, y_0, z_0)$: Point in the surface

$P(x, y, z)$: Arbitrary point in P

$f = f(x,y,z)$

$$\left.\frac{\partial f}{\partial x}\right|_{P_0}(x-x_0) + \left.\frac{\partial f}{\partial y}\right|_{P_0}(y-y_0) + \frac{\partial f}{\partial z}(z-z_0) = 0$$

Simplifies: $\boxed{f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0) = 0}$

NORMAL LINE OF THE SURFACE AT A POINT P_0

$g = x = x_0 + f_x(P_0),\ y = y_0 + f_y(P_0),\ z = z_0 + f_z(P_0)$

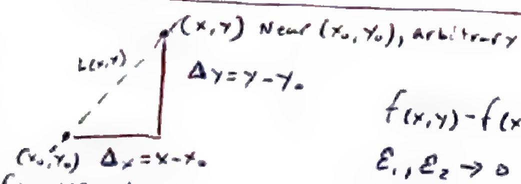
EQUATION OF L(t):

$\boxed{x = x_0 + f_x(P_0)t \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t}$

ESTIMATING CHANGE IN f IN A DIRECTION u

$\boxed{df = (\nabla f|_{P_0} \cdot u)\, ds = (D_u f|_{P_0})\, ds}$ ds = Distance increment per unit

LINEARIZATION OF FUNCTION f (Applies to 2+ vars (append vars))



f is differentiable at this point

$$f(x,y) - f(x_0,y_0) = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

$\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$, so

$$\boxed{f(x,y) \approx \underbrace{f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}_{L(x,y)}}$$

$\boxed{f(x,y) \approx L(x,y) \text{ Standard approximation of } f \text{ at } (x_0, y_0)}$

Error:

$$|E(x,y)| \le \frac{1}{2} M \left(|x-x_0| + |y-y_0|\right)^2$$

M = Upper bound for $|f_{xx}|$, $|f_{yy}|$ and $|f_{xy}|$ on R

TOTAL DIFFERENTIAL (Applies to 2+ vars (append vars))

$$df = f_x(x_0,y_0)\, dx + f_y(x_0,y_0)\, dy$$

dx : Rate of change in the x direction

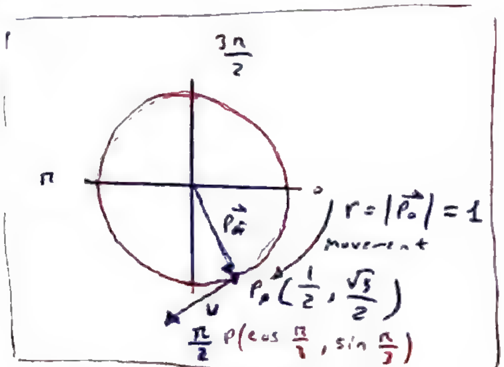
dy : Rate of change in the y direction

25. Temperature along a circle

Suppose that the Celsius temperature at the point (x,y) in the xy-plane is $T(x,y) = x \sin 2y$ and that distance in the xy-plane is measured in meters. A particle is moving clockwise around the circle of radius 1m centered at the origin at the constant rate of 2m/s.

a) How fast is the temperature experienced by the particle changing in degrees celsius per meter at the point $P(\frac{1}{2}, \frac{\sqrt{3}}{2})$?

b) How fast is the temperature experienced by the particle changing in degrees Celsius per second at P?



If P_0 is a position vector, then $|P_0| = 1 = r$. The direction of the particle at any given point in the circle is perpendicular to the direction of the vector $\vec{P_0}$. That is, $u \cdot \vec{P_0} = 0$. u is the direction of the particle at P_0.

The partial derivatives of the function $T(x,y) = x \sin 2x$ are:

$$\frac{\partial T}{\partial x} = \sin 2y \, , \quad \frac{\partial T}{\partial y} = 2x \cos 2y$$

At point P, these are:

$$\left.\frac{\partial T}{\partial x}\right|_{P(\frac{1}{2},\frac{\sqrt{3}}{2})} = \sin 2\frac{\sqrt{3}}{2} = \boxed{\sin\sqrt{3}} \, , \quad \left.\frac{\partial T}{\partial y}\right|_{P(\frac{1}{2},\frac{\sqrt{3}}{2})} = \cancel{2}\frac{1}{\cancel{2}} \cos \cancel{2}\frac{\sqrt{3}}{\cancel{2}} = \boxed{\cos\sqrt{3}}$$

The gradient of the function T at point P is:

$$\boxed{\nabla T|_P = \sin\sqrt{3}\, i + \cos\sqrt{3}\, j}$$

The direction of the particle u is perpendicular to $\vec{P}$, so the i and j components are swapped. The j component points towards the negative area.

$$\boxed{u = \frac{\sqrt{3}}{2} i - \frac{1}{2} j}$$

And the derivative (rate of change) in this direction at point P is:

(a) $$D_u T|_P = \nabla T|_P \cdot u = \boxed{\frac{\sqrt{3}}{2}\sin\sqrt{3} - \frac{1}{2}\cos\sqrt{3} \approx 0'9350684\,°C/m}$$

Finally, the estimated rate of change per second at P is:

b) $$dT = (D_u T|_P)\, ds = \left(\frac{\sqrt{3}}{2}\sin\sqrt{3} - \frac{1}{2}\cos\sqrt{3}\ °C/m\right) \cdot 2\ m/s = \boxed{\sqrt{3}\sin\sqrt{3} - \cos\sqrt{3}}$$

$$\boxed{\approx 1'8701368\,°C/s}$$

1. a) Tangent plane . b) normal line at P_0 on surface

1. $x^2+y^2+z^2=3$ $\quad P_0(1,1,1)$

a)

$\nabla f = 2x\,i + 2y\,j + 2z\,k \quad \nabla f|_{P_0} = 2i+2j+2k$

a) Tangent plane: $2(x-1)+2(y-1)+2(z-1)=0$

$2x-2+2y-2+2z-2=0$

$2(x+y+z)=6$

$\boxed{x+y+z=3}$

b) Normal line

$\boxed{x=1+2t \quad y=1+2t \quad z=1+2t}$

3. $x^2y+2xz^2=8$ $\quad P_0(1,0,2)$

$\nabla f = (2xy+2z^2)\,i + x^2\,j + 4xz\,k$

$\nabla f|_{P_0} = 2(1)(0)+2(2)^2 i + 1j + 4(1)(2)k = 8i+j+8k$

a) Tangent plane: $\boxed{8(x-1)+(y)+8(z-2)=0}$

$8x-8+y+8z-16=0$

$\boxed{8x+y+8z=24}$

b) Normal line

$\boxed{x=1+8t \quad y=t \quad z=2+8t}$

5. $\cos \pi x - x^2y + e^{xz} + yz = 4$ $\quad P_0(0,1,2)$

$-x - x^2y + e^{xz} + yz = 4$

$f_x = -1-2xy+ze^{xz} \quad f_y = \boxed{-x^2+z} \quad f_z = \boxed{xe^{xz}+y}$

$f_x(P_0) = -1-2(0)(1)+(2)e^{(0)(2)} = 2-1 = 1$

$f_y(P_0) = -(0)^2+2 = 2 \qquad f_z(P_0) = (0)e^{(0)(2)}+1 = 1$

$\nabla f|_{P_0} = i+2j+k$

a) $(x-0)+2(y-1)+(z-2)=0$

$x+2y-2+z-2=0$

$\boxed{x+2y+z=4}$

b) $\boxed{\begin{aligned} x &= t \\ y &= 1+2t \\ z &= 2+t \end{aligned}}$

- Find two numbers a and b with $a \leq b$ such that

$$\int_a^b (6 - x - x^2)\, dx$$

has it's largest value.

- Applying Leibnitz's theorem:

$\partial a = -6 + a + a^2$

$\partial b = 6 - b - b^2$

- Applying $\partial a = \partial b = 0$; using the quadratic formula:

$$a = \frac{-1 \pm \sqrt{1 - 4(-1)(6)}}{2(1)} = \frac{-1 \pm 5}{2} = \boxed{\{-3, 3\}}$$

$$b = \frac{1 \pm \sqrt{1 - 4(-1)(6)}}{-2} = \frac{1 \pm 5}{-2} = \boxed{\{-3, 2\}}$$

- The interior points are: $\boxed{\{(3,-3), (3,2), (-3,-3), (-3,2)\}}$. Only two of these points fulfill the requirement $a \leq b$: $\boxed{(-3,-3) \text{ and } (-3,2)}$.
- when $a = -3$ and $b = -3$, $a = b$. The integral is 0.
- when $a = -3$ and $b = 2$:

$$\int (6 - x - x^2)\, dx = 6x - \frac{x^2}{2} - \frac{x^3}{3} + C$$

$$\int_{-3}^{2} (6 - x - x^2)\, dx = 6x - \frac{x^2}{2} - \frac{x^3}{3} \Big|_{-3}^{2} = \left(12 - 2 - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + 9\right) = \boxed{\frac{125}{6} > 0}$$

- when $a \to \infty$ or $a \to -\infty$, $a^2 + a - 6 \to \infty$ $\left(\lim_{a \to -\infty} a^2 + a - 6 = \lim_{a \to -\infty} a(a+1) - 6 = \infty\right)$. when $b \to \infty$, $-b^2 - b + 6 \to -\infty$, and $-b^2 - b + 6 \to \infty$ when $b \to -\infty$. $(-3, 2)$ is a local máximum.

$\delta = 5$ $\quad R: 0 \le x \le 2,\ x \le y \le 2+x^2 \quad dA = dy\,dx$

$m = \iint_R \delta\, dA \qquad M_y = \iint_R x\delta\, dA \qquad M_x = \iint_R y\delta\, dA \qquad \bar{x} = \frac{M_y}{m},\ \bar{y} = \frac{M_x}{M}$

$$m = \int_0^2\int_x^{2+x^2} \delta\, dy\, dx = \delta\int_0^2 y\Big]_x^{2+x^2} dx = \delta\int_0^2 2+x^2-x\, dx = \delta\left[2x + \frac{x^3}{3} - \frac{x^2}{2}\right]_0^2 =$$

$$= \delta\left(4 + \frac{8}{3} - 2\right) = \delta\left(\frac{12}{3} + \frac{8}{3} - \frac{6}{3}\right) = \frac{14}{3}\delta = \boxed{\frac{70}{3}}\ M \qquad [x(2+x^2)] - [x(x)]$$

First moments

$$M_y = \int_0^2\int_x^{2+x^2} x\delta\, dy\, dx = \delta\int_0^2 xy\Big]_x^{2+x^2} dx = \delta\int_0^2 2x + x^3 - x^2\, dx = \delta\left[x^2 + \frac{x^4}{4} - \frac{x^3}{3}\right]_0^2$$

$$= \delta\left(4 + 4 - \frac{8}{3}\right) = \delta\left(\frac{24}{3} - \frac{8}{3}\right) = \delta\,\frac{16}{3} = \frac{80}{3}\ M_y$$

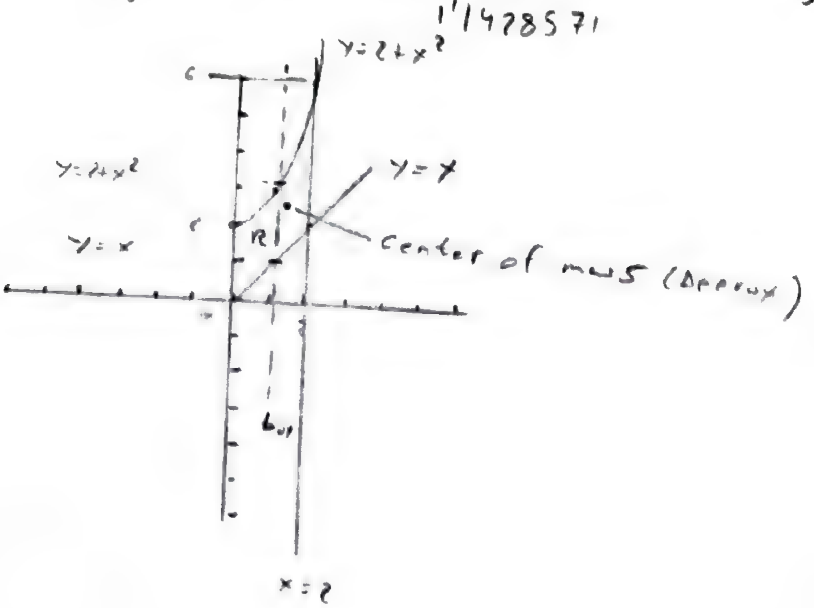
$$M_x = \int_0^2\int_x^{2+x^2} y\delta\, dy\, dx = \delta\int_0^2 \frac{y^2}{2}\Big]_x^{2+x^2} dx = \delta\int_0^2 \frac{(2+x^2)^2}{2} - \frac{x^2}{2}\, dx =$$

$$= \delta\left[2x + \frac{2x^3}{3} + \frac{x^3}{10} - \frac{x^3}{6}\right]_0^2 = \delta\left(4 + \frac{16}{3} + \frac{32}{10} - \frac{8}{6}\right) = \delta\,\frac{168}{15} = \frac{840}{15} = 56\ M_x$$

Center of mass

$$\bar{x} = \frac{M_y}{m} = \frac{\frac{80}{3}}{\frac{70}{3}} = \frac{240}{210} = \frac{24}{21} = \boxed{\frac{8}{7}} \qquad \bar{y} = \frac{M_x}{m} = \frac{56}{\frac{70}{3}} = \frac{168}{70} \approx \boxed{2'4}$$

1'1428571

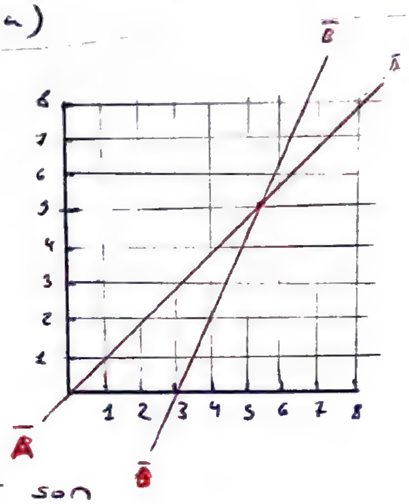


Línea recta

Pendiente $= m = \dfrac{y_2 - y_1}{x_2 - x_1}$ | Ecuación de la pendiente: $(y_2 - y_1) = m(x_2 - x_1)$ (Línea)

- Cálculo de la intersección entre $\bar{A}$ y $\bar{B}$:

Línea $\bar{A}$: (0,0), (8,8)
Línea $\bar{B}$: (3,0), (6,7)



- Ecuación de $\bar{A}$: $\boxed{m\bar{A} = \frac{8-0}{8-0} = \frac{8}{8} = 1}$

$(y-0) = 1(x-0)$

$\boxed{\bar{A}: y = x}$

- Ecuación de $\bar{B}$: $\boxed{m\bar{B} = \frac{7-0}{6-3} = \frac{7}{3}}$

$y - 0 = \frac{7}{3}(x-3)$

$\boxed{\bar{B}: y = \frac{7}{3}(x-3)}$

- Se igualan las ecuaciónes de ambas rectas: $(\bar{A} = \bar{B})$, y se resuelve para x. Las soluciones son las intersecciónes entre $\bar{A}$ y $\bar{B}$.

$x = \frac{7}{3}(x-3)$

$x = \frac{7(x-3)}{3}$

$3x = 7x - 21$

$4x - 21 = 0$

$4x = 21$

$\boxed{x = \frac{21}{4}}$

- La intersección entre $\bar{A}$ y $\bar{B}$ se encuentra en el punto $x = \frac{21}{4}$

$\bar{A}(\frac{21}{4}) = \frac{21}{4} \approx 5'25$

$\bar{B}(\frac{21}{4}) = \frac{7}{3}(\frac{21}{4} - 3) = \frac{7}{3} \cdot \frac{9}{4} = \frac{63}{12} \approx 5'25$

$\boxed{\bar{A} \text{ y } \bar{B} \text{ son iguales en el punto } x = \frac{21}{4}}$

Colisión de objetos

- Un tren sale de una estación (A), a una velocidad de 120 Km/h. Mientras, otro tren sale de una estación (B) a 100 Km de distancia, y a una velocidad de 90 Km/h, en sentido contrario por la vía opuesta. Calcular el punto en el que se cruzan y el momento en el que lo hacen, asumiendo velocidades constantes.

A — 100Km — B

- Pasar las unidades al SI: $\boxed{V_A = 120\left(\frac{1}{3'6}\right) = 33'\hat{3}\ m/s}$ $\boxed{V_B = 90\left(\frac{1}{3'6}\right) = 25\ m/s}$ $\boxed{100Km = 100.000m}$

- La posición inicial de cada tren es la condición inicial

$\boxed{S_{(0)}A = 0}$, $\boxed{S_{(0)}B = 100.000}$

- La ~~Posición~~ Velocidad es la primera derivada de la posición. Integramos para calcular la ecuación de la posición de cada tren.

$S_A = \int 33'\hat{3}\,dt = 33'\hat{3}t + C$. Como $S_{(0)}A = 0$, $C = 0$. Por tanto, $\boxed{S_A = 33'\hat{3}t}$

$S_B = -\int 25\,dt = -25t + C$. $S_{(0)}B = 100.000$. $-25(0) + C = 100.000$. $C = 100.000$. Por tanto, $S_B = -25t + 100.000$ o $\boxed{S_B = 100.000 - 25t}$

○ Tiempo de colisión (cruce en éste caso): $S_A = S_B$ y resolver para x

$33'\hat{3}t = 100.000 - 25t \Rightarrow 58'\hat{3}t - 100.000 = 0 \Rightarrow \boxed{t = \frac{100.000}{58'\hat{3}} = 1714'2857 \text{ segundos}}$ $\frac{1714'2857}{60} = 28'5714$ minutos

○ Posición de la colisión (cruce):

$S_{(1714'2857)}A = S_{(1714'2857)}B = 33'\hat{3}(1714'2857) = 57142'8575\,m = \boxed{57'142\,Km}$

La colisión se produce en el punto 57'142Km en $t = 1714'2857$ (Alrededor de 28:30 minutos).

FÓRMULA DE TAYLOR

La función o fórmula de Taylor nos da una aproximación de una función en términos de las derivadas de la función.

Por ejemplo, si $f(x) = \sin x$, $f^{(1)}(x) = \cos x$, $f^{(2)}(x) = -\sin x$, $f^{(3)}(x) = -\cos x$. Y así sucesivamente. En el caso de e^x es incluso más sencillo, ya que la derivada de e^x es e^x.

La función $f(x)$ se puede representar como $f^{(0)}(x)$. De modo que $f(x) = f^{(0)}(x)$.

Factorial

El factorial de un entero n se representa como $n!$. Es la multiplicación sucesiva de todos los enteros de 1 hasta n.

$1! = 1 \quad 2! = 2 \quad 3! = 6 \ (1 \cdot 2 \cdot 3) \quad 4! = (1 \cdot 2 \cdot 3 \cdot 4) = 24$

Y así sucesivamente. El factorial de 0 es 1 $\boxed{(0! = 1)}$. En general,

$$\boxed{n! = n(n-1)(n-2)\cdots 2 \cdot 1}$$

Polinomio de la fórmula de Taylor

$$\boxed{P(x) = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n}$$

Los números $C_0, \ldots, C_n$ se llaman coeficientes del polinomio. Estos coeficientes se pueden expresar en términos de las derivadas de $P(x)$ en el punto $x = 0$.

Siendo K un íntegro ≥ 0, la derivada K de $P(x)$ se da por la fórmula:

$$\boxed{P^{(K)}(x) = C_K K! + \text{Expresión que contiene un factor de } x.}$$

La razón es que si derivamos K veces los términos

$$C_0, C_1 x, \ldots, C_K x^{K-1}$$

obtendremos 0. Y si derivamos K veces una potencia x^j con $j > K$ entonces alguna potencia positiva de x será el resultado. Si evaluamos la K-ésima ~~potencia~~ derivada en el punto $x = 0$, obtenemos:

$$\boxed{P^{(K)}(0) = C_K K!}$$

Ya que sustituimos las x por 0. Por tanto, podemos despejar C_K dividiendo por $K!$ a ambos lados para obtener:

$$\boxed{C_K = \frac{P^{(K)}(0)}{K!}}$$

Ahora, siendo f una función que es derivable hasta el orden n en un intervalo. Buscamos un polinomio

$$P(x) = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$$

cuyas derivadas en 0 (hasta el orden n) sean las mismas que las de la función f en el punto $x = 0$. En otras palabras:

$$\boxed{P^{(K)}(0) = f^{(K)}(0)}$$

(Siendo P el polinomio y f la función original).

¿Cuáles deben ser los coeficientes $c_0, \dots, c_n$ para conseguirlo? La respuesta es obvia dada la fórmula que hemos obtenido.

Si $P^{(k)}(0) = f^{(k)}(0)$ y $c_k k! = P^{(k)}(0)$, entonces:

$$k!\, c_k = f^{(k)}(0)$$

Para todo integro $k = 0, 1, \dots, n$. Despejamos c_k para obtener la expresión:

$$\boxed{c_k = \frac{f^{(k)}(0)}{k!}}$$ Fórmula del k-ésimo coeficiente de $P(x)$

Por tanto, el Polinomio de Taylor de grado $\leq n$ para la función f es el polinomio:

$$\boxed{P_{(n)}(x) = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n}$$

Ejemplo:

Sea $f(x) = \sin x$. Los polinomios de Taylor tienen la fórmula:

$$\boxed{P_{2m+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^m \frac{x^{2m+1}}{(2m+1)!}}$$

Hacemos que n sea $2m+1$ ya que cuando $f(x) = \sin x$, el valor de $f^{(k)}(0) = \pm 1$ sólo en los valores impares.

$k=0$ $\sin 0 = 0$, $k=1$ $\cos 0 = 1$, $k=2$ $-\sin 0 = 0$, $k=3$ $-\cos 0 = -1$, $k=4$ $\sin 0 = 0$, y así sucesivamente.

$2m+1$ de la sucesión $1, 3, 5, 7, \dots, 2m+1$. Por otro lado, la función es negativa en los valores $3, 7$, etc. Por tanto, se multiplica por $(-1)^m$ para cambiar el signo.

Ejercicio: Encontrar el Polinomio de Taylor de $f(x) = \cos x$

La función $\cos x$ tiene infinitas derivadas $f^{(n)}(x) = \frac{d^n}{d^n x}\cos x$. Cuando n es impar, en el punto $x=0$, la derivada es 0 ($f^{(2n+1)}(0) = 0$). Y 1 cuando es par ($f^{(2n)}(0) = \pm 1$). Por último, el valor alterna positivo-negativo-positivo en los pares.

$f^{(0)}(0) = \cos 0 = 1$, $f^{(1)}(0) = -\operatorname{sen} 0 = 0$, $f^{(2)}(0) = -\cos 0 = -1$, $f^{(3)}(0) = \operatorname{sen} 0 = 0$, $f^{(4)}(0) = \cos 0 = 1 \dots$

La fórmula es:

$$\boxed{P_n(x) = (-1)^n \frac{x^{2n}}{(2n)!}}$$

Cuyo resultado es:

$$\boxed{P_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots + (-1)^n\left(\frac{x^{2n}}{(2n)!}\right)}$$

O lo que es lo mismo.

$$\boxed{P_n(x) = \sum_{i=0}^{n} (-1)^i \frac{x^{2i}}{(2i)!}}$$

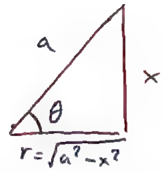
$x = a \tan\theta$

$dx = a \sec^2\theta \, d\theta$

$\theta = \arctan \frac{x}{a}$

$r = \frac{a}{\cos\theta} = a|\sec\theta|$

$a = r\cos\theta$



$x = a \sin\theta$

$dx = a \cos\theta \, d\theta$

$\theta = \arcsin \frac{x}{a}$

$r = a|\cos\theta|$

$a = r \sec\theta$

$x = a \sec\theta$

$dx = a \sec\theta \tan\theta \, d\theta$

$\theta = \operatorname{arc\,sec} \frac{x}{a}$

$r = a|\tan\theta|$

$a = r \cot\theta$

$$\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-a^2\sin^2\theta}} = a\int \frac{\cos\theta\, d\theta}{\sqrt{a^2(1-\sin^2\theta)}} = a\int \frac{\cancel{\cos\theta}\, d\theta}{a\,\cancel{\cos\theta}} = a\int \frac{d\theta}{a} = \theta + C = \arcsin\frac{x}{a} + C = \boxed{\arcsin\frac{x}{2} + C}$$

$a = \sqrt{4} = 2$

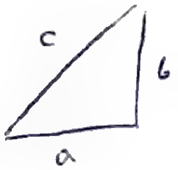
Factor $a^2 - a^2\sin^2\theta$

$1-\sin^2\theta = \cos^2\theta$

$a\frac{1}{a} = 1$

- Express x, r in terms of a and θ
- Express θ in terms of $\frac{x}{a}$
- $\sqrt{x^2-a^2}$ is $\sqrt{a^2-x^2}$, but in reverse (The opposite angle)

If $a > 0$



$c^2 = a^2 + b^2$

$a^2 = c^2 - b^2$

$b^2 = c^2 - a^2$

$\tan\theta = \frac{x}{a}$

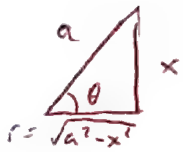
$x = a \tan\theta$

$dx = a \sec^2\theta \, d\theta$

$\theta = \arctan \frac{x}{a}$

$a = r\cos\theta$

$r = \frac{a}{\cos\theta} = a|\sec\theta|$ $(\tan\theta)$



$\sin\theta = \frac{x}{a}$

$x = a \sin\theta$

$dx = a \cos\theta \, d\theta$

$\theta = \arcsin \frac{x}{a}$

$r = a|\cos\theta|$ $\cos\theta = \frac{r}{a}$

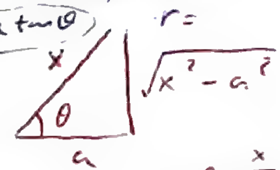
$\cancel{a = x \sin\theta}$

$\cancel{x = \frac{a}{\sin\theta}}$

$a = r\cot\theta$ $\cancel{r = x\csc\theta}$

$r = \frac{a}{\cot\theta} = a\tan\theta$

$r =$

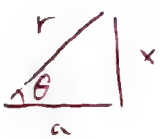


$\sec\theta = \frac{x}{a}$

$x = a \sec\theta$

$dx = a \sec\theta \tan\theta \, d\theta$

$\theta = \operatorname{arcsec} \frac{x}{a}$ $r = a|\tan\theta|$



$r^2 = a^2 + x^2$

$\boxed{r = \sqrt{a^2+x^2}}$ $\rightarrow$

$\sin\theta = \frac{x}{r}$

$\boxed{\begin{array}{l} x = r\sin\theta \\ x = \sqrt{a^2+x^2}\,\sin\theta \end{array}}$

$x^2 = (a^2+x^2)\sin^2\theta$

$x^2 - x^2\sin^2\theta = a^2\sin^2\theta$

$x^2\cos^2\theta = a^2\sin^2\theta$

$x^2 = \frac{a^2\sin^2\theta}{\cos^2\theta} = a^2\tan^2\theta$

$\boxed{x = a\tan\theta}$

$a^2 = \frac{x^2\cos^2\theta}{\sin^2\theta} \rightarrow \boxed{a = x\cot\theta}$

	$d\theta$	$\int d\theta$
$\sin\theta$	$\cos\theta$	$-\cos\theta + C$
$\cos\theta$	$-\sin\theta$	$\sin\theta + C$
$\frac{\sin\theta}{\cos\theta}$ $\tan\theta$	$\sec^2\theta$	$-\ln\lvert\cos\theta\rvert + C$
$\frac{\cos\theta}{\sin\theta}$ $\cot\theta$	$-\csc^2\theta$	$\ln\lvert\sin\theta\rvert + C$
$\frac{1}{\cos}$ $\sec\theta$	$\sec x \tan x$ or $\frac{\tan x}{\cos x}$	$\ln\lvert\sec\theta + \tan\theta\rvert + C$
$\frac{1}{\sin}$ $\csc\theta$	$-\csc x \cot x$ or $-\frac{\cot x}{\sin x}$	$-\ln\lvert\csc\theta + \cot\theta\rvert + C$

$\int u \, dv = uv - \int v \, du + C$

$$\frac{d}{dx}\left[\sin^2 x\right] = \boxed{2\sin x\cos x}$$

$$\frac{d}{dx}\left[\cos^2 x\right] = \boxed{-2\sin x\cos x}$$

$\sin^2 x + \cos^2 x = 1$

$$\frac{d}{dx}\left[\tan^2 x\right] = \frac{d}{dx}\left[\frac{\sin^2 x}{\cos^2 x}\right] = \frac{2\sin x\cos^3 x + 2\sin^3 x\cos x}{\cos^4 x} = \frac{2\sin x\cos x(\overline{\sin^2 x+\cos^2 x})}{\cos^4 x} =$$

$$= \boxed{\frac{2\sin x}{\cos^3 x}}$$

$\sin^2 x + \cos^2 x = 1$

$$\frac{d}{dx}\left[\cot^2 x\right] = \frac{d}{dx}\left[\frac{\cos^2 x}{\sin^2 x}\right] = -\frac{2\sin^3 x\cos x - 2\sin x\cos^3 x}{\sin^4 x} = -\frac{2\sin x\cos x(\overline{\sin^2 x+\cos^2 x})}{\sin^4 x} = \boxed{-\frac{2\cos x}{\sin^3 x}}$$

$$\frac{d}{dx}\left[\sec^2 x\right] = \frac{d}{dx}\left(\frac{1}{\cos^2 x}\right) = \frac{2\sin x\cos x}{\cos^4 x} = \boxed{\frac{2\sin x}{\cos^3 x} \text{ or } \frac{2\tan x}{\cos^2 x}}$$

$$\frac{d}{dx}\left[\csc^2 x\right] = \frac{d}{dx}\left(\frac{1}{\sin^2 x}\right) = -\frac{2\sin x\cos x}{\sin^4 x} = \boxed{-\frac{2\cos x}{\sin^3 x} \text{ or } -\frac{2\cot x}{\sin^2 x}}$$

Euler e: $\boxed{e = \left(1+\frac{1}{n}\right)^n}$

Odds: $O(x) = \frac{\text{Tries}}{\text{Max tries}}$ $\left(\frac{1}{2}, 0\text{'s} \ldots\right)$ $\boxed{O(x) = \frac{P(x)}{1-P(x)}}$

Probability: $\boxed{P(x) = \frac{O(x)}{1+O(x)}}$ $0 \le P(x) \le 1$

$\boxed{P(A \text{ AND } B) = P(A) \times P(B)}$ Conditional: $P(A \text{ GIVEN } B)$ or $P(A|B)$

$\boxed{P(A \text{ OR } B) = P(A) + P(B)}$ (Mutually exclusive only)

Sum rule of probability: (~~Also~~ works when A and B are mutually and non-mutually exclusive)

$P(A \text{ OR } B) = P(A) + P(B) - P(A \text{ AND } B) = \boxed{P(A) + P(B) - P(A) \times P(B)}$

Baye's Theorem: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Fiesor Probability

A: Coffe drinker B: Cancer

P(A|B): Probability of being coffee drinker having cancer

P(B|A): Probability of having cancer being a coffee drinker

Área de un disco de radio r

$x = r\cos\theta$

$y = r\sin\theta$

Circumferencia $C = 2\pi r$

Integración simple

$$A_D = \int C\,dr = \int 2\pi r\,dr = 2\pi\frac{r^2}{2} + c = \pi r^2 + c$$

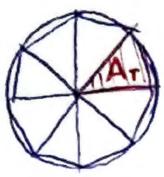
Integración doble

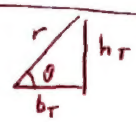
Sea R una constante con el valor del radio,

$$A_D = \int_0^{2\pi}\int_0^R r\,dr\,d\theta = \int_0^{2\pi}\left.\frac{r^2}{2}\right]_0^R d\theta = \int_0^{2\pi}\frac{R^2}{2}\,d\theta = \frac{R^2}{2}\Big[\theta\Big]_0^{2\pi} = 2\pi\frac{R^2}{2} = \pi R^2$$

(πr^2)

Integración por aproximación geométrica





$b_T = r\cos\frac{2\pi}{n}$ $\qquad \theta_T = \frac{2\pi}{n}$

$h_T = r\sin\frac{2\pi}{n}$

D es un disco dividido en n triángulos de igual tamaño. Cada triángulo A_T tiene un área $\frac{b_T h_T}{2}$, y el ángulo de cada triángulo es $\theta_T = \frac{2\pi}{n}$. Es decir, si el círculo exterior a D tiene 2π rad, cada triángulo tiene $\frac{2\pi}{n}$ rad. Por tanto:

$$\boxed{A_T = \frac{b_T h_T}{2} = \frac{r^2\sin\frac{2\pi}{n}\cos\frac{2\pi}{n}}{2}}$$

~~El área de D se~~

Según incrementamos el número de triángulos (n), el área del polígono que forman los triángulos se aproxima al área del disco. Por tanto:

$$\boxed{A_D = \lim_{n\to\infty} n A_T = \lim_{n\to\infty} n\frac{r^2\sin\frac{2\pi}{n}\cos\frac{2\pi}{n}}{2} = \frac{2\pi r^2}{2} = \pi r^2}$$

Cálculo del límite

Regla del producto

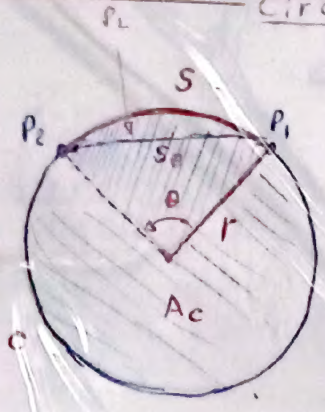
- $\lim_{n\to\infty} nr^2\frac{\sin(\frac{2\pi}{n})\cos(\frac{2\pi}{n})}{2} = \left(\frac{1}{2}r^2\right)\left(\lim_{n\to\infty} n\sin(\frac{2\pi}{n})\right)\left(\lim_{n\to\infty}\cos(\frac{2\pi}{n})\right)$
- $\lim_{n\to\infty} n\sin(\frac{2\pi}{n})$ L'hopital: $\lim_{n\to\infty}\frac{-\frac{2\pi}{n^2}\cos(\frac{2\pi}{n})}{-\frac{1}{n^2}}$ * $= \boxed{\lim_{n\to\infty}\cancel{n^2}\frac{2\pi}{\cancel{n^2}}\cos(\frac{2\pi}{n}) = 2\pi}$
- $\lim_{n\to\infty}\cos(\frac{2\pi}{n}) = 1$ **
- $\therefore \lim_{n\to\infty} nr^2\frac{\sin(\frac{2\pi}{n})\cos(\frac{2\pi}{n})}{2} = \frac{1}{\cancel{2}}r^2\cancel{2}\pi(1) = \boxed{\pi r^2}$

* Usando $n\sin(\frac{2\pi}{n}) = \frac{\sin(\frac{2\pi}{n})}{\frac{1}{n}}$ ya que $\frac{1}{\frac{1}{n}} = n$

** $\lim_{n\to\infty}\frac{2\pi}{n} = 0$

$$\boxed{C = \lim_{n\to\infty}\sum^n r\sin(\tfrac{2\pi}{n}) = r\lim_{n\to\infty} n\sin(\tfrac{2\pi}{n}) = 2\pi r}$$

Círculo



Circunferencia (c)

$$C = 2\pi r$$

Área del círculo (Ac)

$$\int c\,dr = \pi r^2$$

Longitud de arco (S)

$$S = \theta r$$ (ángulo por radio)

Ángulo (θ)

$$\theta = \frac{S}{r}$$

Radio (r)

$$r = \frac{S}{\theta}$$

Área del sector (Se)

$$Se = \frac{\theta}{2} r^2$$ (mitad del ángulo por radio al cuadrado)

Distancia entre P_1 y P_2 (P_L)

$$P_L = \sqrt{2r^2} = r\sqrt{2}$$

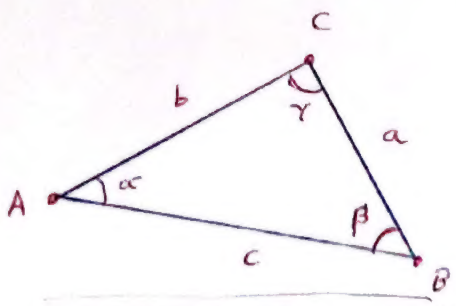
Radianes

$$1\,rad = \left(\frac{180}{\pi}\right)^\circ$$

$$1^\circ = \left(\frac{\pi}{180}\right) rad$$

(conversión)

Cualquier Triángulo



$$\alpha + \beta + \gamma = \pi\ rad = 180^\circ$$

a es el lado opuesto a α (A)
b es el lado opuesto a β (B)
c es el lado opuesto a γ (C)

Regla de los senos

$$\frac{\operatorname{sen}\alpha}{a} = \frac{\operatorname{sen}\beta}{b} = \frac{\operatorname{sen}\gamma}{c}$$

$$\frac{a}{\operatorname{sen}\alpha} = \frac{b}{\operatorname{sen}\beta} = \frac{c}{\operatorname{sen}\gamma}$$

Se pueden invertir (son iguales)

Regla de los cosenos

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

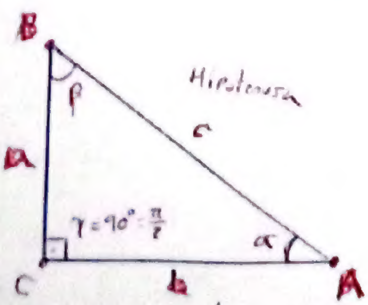
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc} \qquad \cos\gamma = \frac{a^2+b^2-c^2}{2ab}$$

$$\cos\beta = \frac{a^2+c^2-b^2}{2ac}$$

Triángulo Rectángulo



Opuesto (a A)
Adyacente (a B)

Adyacente (a A)
Opuesto (a B)

Relaciones α

$$\operatorname{sen}\alpha = \frac{opuesto}{hip} = \frac{a}{c}$$

$$\cos\alpha = \frac{Adyacente}{hip} = \frac{b}{c}$$

$$\tan\alpha = \frac{opuesto}{adyacente} = \frac{a}{b}$$

Relaciones β

$$\operatorname{sen}\beta = \frac{opuesto}{hip} = \frac{b}{c}$$

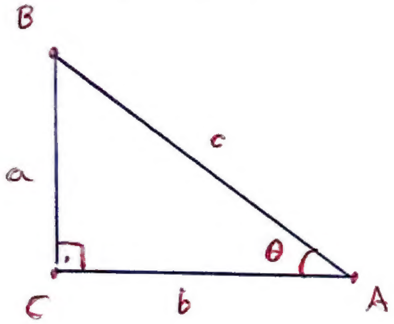
$$\cos\beta = \frac{ady}{hip} = \frac{a}{c}$$

$$\tan\beta = \frac{opuesto}{ady} = \frac{b}{a}$$

Ley de Pitágoras (Teorema)

$$c^2 = a^2 + b^2$$

$$c = \sqrt{a^2+b^2}$$



Relaciones Trigonométricas

$\operatorname{sen} \theta = \frac{a}{c}$ $\qquad \cot \theta = \frac{1}{\tan \theta} = \frac{b}{a}$

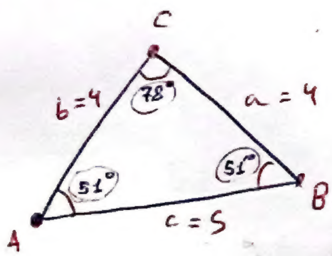
$\cos \theta = \frac{b}{c}$ $\qquad \sec \theta = \frac{1}{\cos \theta} = \frac{c}{b}$

$\tan \theta = \frac{a}{b}$ $\qquad \csc \theta = \frac{1}{\operatorname{sen} \theta} = \frac{c}{a}$

$a = c \sin \theta = b \tan \theta$

$b = c \cos \theta = a \cot \theta$

$c = b \sec \theta = a \csc \theta$



- $\cos A = \cos B = \frac{b^2+c^2-a^2}{2bc} = \frac{16+25-16}{2 \cdot 4 \cdot 5} = \frac{25}{40} = \frac{5}{8}$
- $A = B = \arccos\left(\frac{5}{8}\right) = 0'8956 \text{ rad}$
- $C = \pi - 2 \cdot 0'8956 = 1'3503 \text{ rad}$
- $A^\circ = B^\circ = \frac{180}{\pi} 0'8956 \text{ rad} \approx 51^\circ$
- $C^\circ = \frac{180}{\pi} 1'3503 \text{ rad} \approx 78^\circ$
